Modern Data Assimilation for Numerical Weather Prediction

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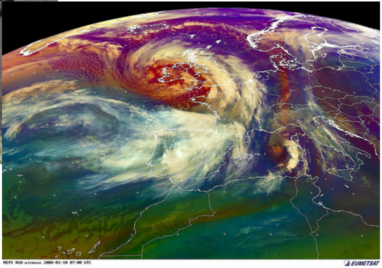
Reading
Nov 12, 2014

Weather is Relevant I ...

Warn and Protect

Plan Travel



Weather is Relevant II ...

Logistics

Rivers and Environment

Air Control

Outline

Numerical Weather Prediction and DWD

- Can Numerics Help?
- Research and Development at DWD

Dynamical Systems, Inverse Problems and Data Assimilation

- Fluid Dynamics and Micro- and Macro-Physics
- Measurements: Stations, Sondes, Planes, Satellites

Data Assimilation Methods

- Tikhonov Regularization and 3dVar
- 4dVar
- Kalman Filter: Deterministic and Stochastic View
- Ensemble Kalman Filter
- Local Ensemble Transform Kalman Filter (LETKF)

Challenges and Open Questions

Outline

Outline

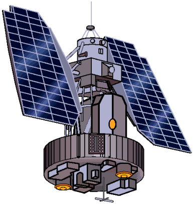
Remarks on the History of Weather Prediction I

- In 1901 Cleveland Abbe it the founder of the United States Weather Bureau. He suggested that the atmosphere followed the principles of thermodynamics and hydrodynamics
- In 1904, Vilhelm Bjerknes proposed a two-step procedure for model-based weather forecasting. First, a analysis step of data assimilation to generate initial conditions, then a forecasting step solving the initial value problem.
- In 1922, Lewis Fry Richardson carried out the first attempt to perform the weather forecast numerically.
- In 1950, a team of the American meteorologists Jule Charney, Philip Thompson, Larry Gates, and Norwegian meteorologist Ragnar Fjörtoft and the applied mathematician John von Neumann, succeeded in the first numerical weather forecast using the ENIAC digital computer.

Bjerknes

Remarks on the History of Weather Prediction II

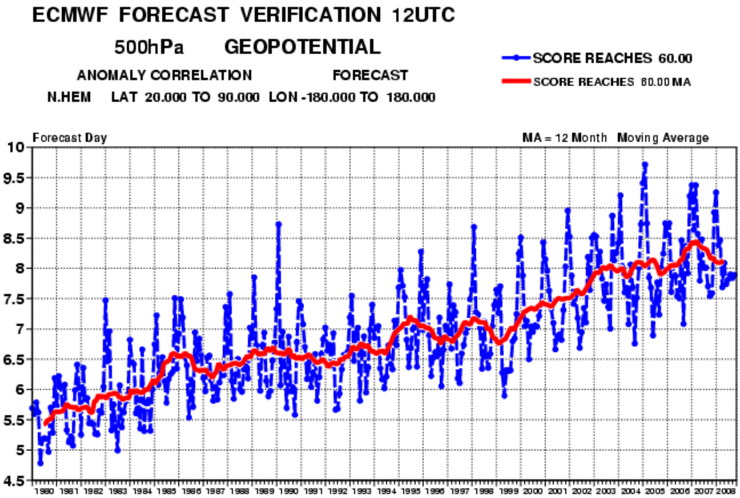
1962



Nimbus 1: 1964

- In September 1954, Carl-Gustav Rossby's group at the Swedish Meteorological and Hydrological Institute produced the first operational forecast (i.e. routine predictions for practical use) based on the barotropic equation. Operational numerical weather prediction in the United States began in 1955 under the Joint Numerical Weather Prediction Unit (JNWPU), a joint project by the U.S. Air Force, Navy, and Weather Bureau.
- In 1959, Karl-Heinz Hinkelmann produced the first reasonable primitive equation forecast, 37 years after Richardson's failed attempt. Hinkelmann did so by removing high-frequency noise from the numerical model during initialization.
- In 1966, West Germany and the United States began producing operational forecasts based on primitive-equation models, followed by the United Kingdom in 1972, and Australia in 1977.

Skills and Scores



Outline

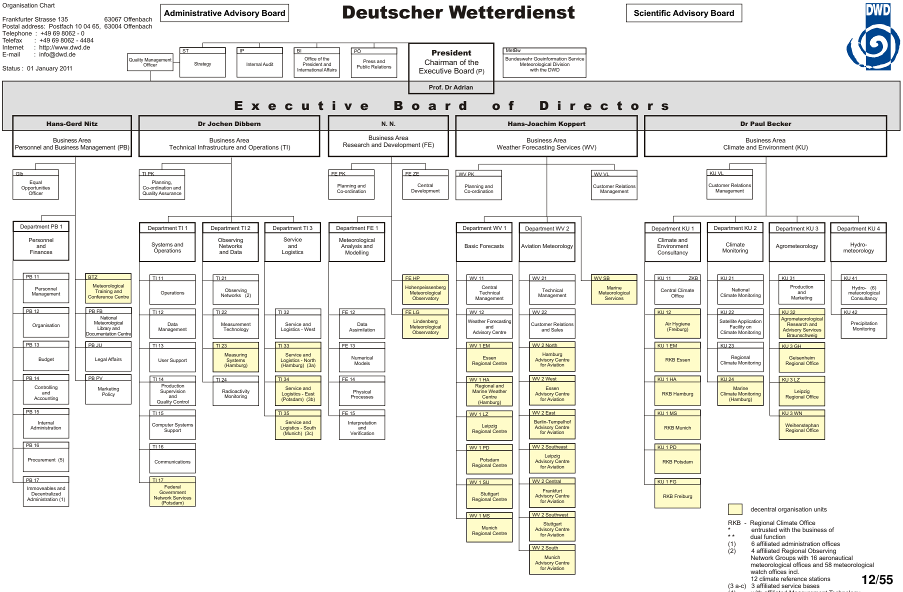
Organizational Structure DWD

Research and Development

- Section on Modelling
  - Unit Num. Modelling
  - Unit Data Assimilation
  - Unit Physics
  - Unit Verification
- Central Development
  - Visualization
  - Products
  - Model Output Statistics
- Meteorological Observatory Lindenberg
- Meteorological Observatory Hohenpeissenberg

DWD Business Areas

- Research and Development
- Climate and Environment
- Human Ressources
- Weather Forecast
- Technical Infrastructure


Organisation Chart
Frankfurter Strasse 135 63067 Offenbach
Postal address: Postfach 10 04 65, 63004 Offenbach
Telephone : +49 69 8062 - 0
Telefax : +49 69 8062 - 4484
Internet : http://www.dwd.de
E-mail : info@dwd.de
Status : 01 January 2011
Administrative Advisory Board
Deutscher Wetterdienst
Scientific Advisory Board
DWD
Quality Management Officer
ST Strategy
IP Internal Audit
BI Office of the President and International Affairs
PO Press and Public Relations
President
Chairman of the Executive Board (P)
Prof. Dr Adrian
MetBw Bundeswehr Goeinformation Service Meteorological Division with the DWD
Executive Board of Directors
Hans-Gerd Nitz
Business Area Personnel and Business Management (PB)
Dr Jochen Dibbern
Business Area Technical Infrastructure and Operations (TI)
N. N.
Business Area Research and Development (FE)
Hans-Joachim Koppert
Business Area Weather Forecasting Services (WV)
Dr Paul Becker
Business Area Climate and Environment (KU)
GIb Equal Opportunities Officer
TI PK Planning, Co-ordination and Quality Assurance
FE PK Planning and Co-ordination
FE ZE Central Development
WV PK Planning and Co-ordination
WV VL Customer Relations Management
KU VL Customer Relations Management
Department PB 1 Personnel and Finances
PB 11 Personnel Management
PB 12 Organisation
PB 13 Budget
PB 14 Controlling and Accounting
PB 15 Internal Administration
PB 16 Procurement (5)
PB 17 Immoveables and Decentralized Administration (1)
BTZ Meteorological Training and Conference Centre
PB FB National Meteorological Library and Documentation Centre
PB JU Legal Affairs
PB PV Marketing Policy
Department TI 1 Systems and Operations
TI 11 Operations
TI 12 Data Management
TI 13 User Support
TI 14 Production Supervision and Quality Control
TI 15 Computer Systems Support
TI 16 Communications
TI 17 Federal Government Network Services (Potsdam)
Department TI 2 Observing Networks and Data
TI 21 Observing Networks (2)
TI 22 Measurement Technology
TI 23 Measuring Systems (Hamburg)
TI 24 Radioactivity Monitoring
Department TI 3 Service and Logistics
TI 32 Service and Logistics - West
TI 33 Service and Logistics - North (Hamburg) (3a)
TI 34 Service and Logistics - East (Potsdam) (3b)
TI 35 Service and Logistics - South (Munich) (3c)
Department FE 1 Meteorological Analysis and Modelling
FE 12 Data Assimilation
FE 13 Numerical Models
FE 14 Physical Processes
FE 15 Interpretation and Verification
FE HP Hohenpeissenberg Meteorological Observatory
FE LG Lindenberg Meteorological Observatory
Department WV 1 Basic Forecasts
WV 11 Central Technical Management
WV 12 Weather Forecasting and Advisory Centre
WV 1 EM Essen Regional Centre
WV 1 HA Regional and Marine Weather Centre (Hamburg)
WV 1 LZ Leipzig Regional Centre
WV 1 PD Potsdam Regional Centre
WV 1 SU Stuttgart Regional Centre
WV 1 MS Munich Regional Centre
Department WV 2 Aviation Meteorology
WV 21 Technical Management
WV 22 Customer Relations and Sales
WV 2 North Hamburg Advisory Centre for Aviation
WV 2 West Essen Advisory Centre for Aviation
WV 2 East Berlin-Tempelhof Advisory Centre for Aviation
WV 2 Southeast Leipzig Advisory Centre for Aviation
WV 2 Central Frankfurt Advisory Centre for Aviation
WV 2 Southwest Stuttgart Advisory Centre for Aviation
WV 2 South Munich Advisory Centre for Aviation
WV SB Marine Meteorological Services
Department KU 1 Climate and Environment Consultancy
KU 11 ZKB Central Climate Office
KU 12 Air Hygiene (Freiburg)
KU 1 EM RKB Essen
KU 1 HA RKB Hamburg
KU 1 MS RKB Munich
KU 1 PD RKB Potsdam
KU 1 FG RKB Freiburg
Department KU 2 Climate Monitoring
KU 21 National Climate Monitoring
KU 22 Satellite Application Facility on Climate Monitoring
KU 23 Regional Climate Monitoring
KU 24 Marine Climate Monitoring (Hamburg)
Department KU 3 Agrometeorology
KU 31 Production and Marketing
KU 32 Agrometeorological Research and Advisory Services Braunschweig
KU 3 GH Geisenheim Regional Office
KU 3 LZ Leipzig Regional Office
KU 3 WN Weihenstephan Regional Office
Department KU 4 Hydro-meteorology
KU 41 Hydro- (6) meteorological Consultancy
KU 42 Precipitation Monitoring
decentral organisation units
RKB - Regional Climate Office
* entrusted with the business of
* * dual function
(1) 6 affiliated administration offices
(2) 4 affiliated Regional Observing Network Groups with 16 aeronautical meteorological offices and 58 meteorological watch offices incl. 12 climate reference stations
(3 a-c) 3 affiliated service bases

Operational Center with Supercomputers

Development Units: FE1, FE12 (Data Assimilation)

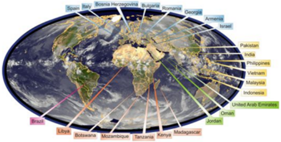
Around 50-60 Scientists on Numerical Modelling

Research > Development > Coding > Operation > Monitoring

National and International Network

Max Planck Institute Meteorologie Hamburg, GFZ Potsdam, Alfred Wegner Institute Bremerhafen, DLR Oberpfaffenhofen, KIT (Karlsruhe Institute of Technology), Universities in Bremen, Cologne, Bonn, Göttingen, Reading, Postsdam, Munich, Berlin, ...

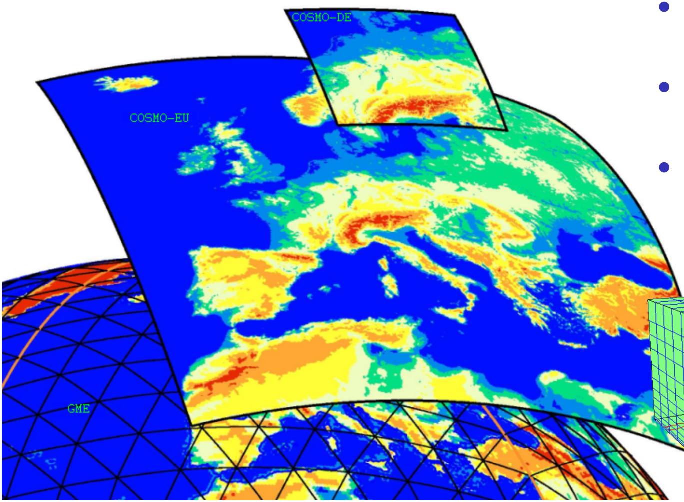
COSMO Consortium



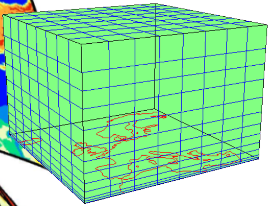
Outline

Outline

Modelling of the Atmosphere: Geometry



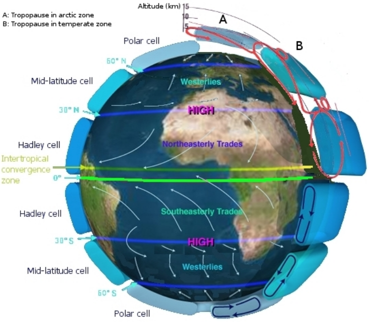
- GME/ICON
  Resolution 20km/13km
- COSMO-EU
  Resolution 7km/6.5km
- COSMO-DE
  Resolution
  2.8km/2.2km



Fluid Dynamics, Winds, Radiation, Heat, Rain, Clouds, Aerosols

Differential Equtions/ Primitive Equations

- Conservation of momentum
- Thermal energy equations
- Continuity equations: conservation of mass

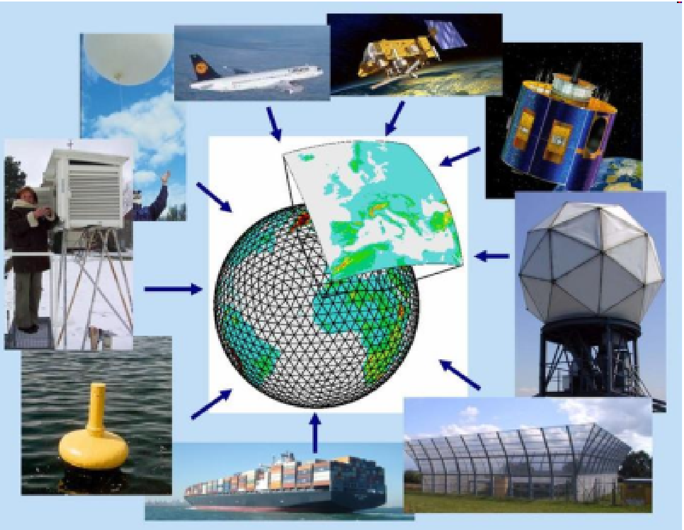


Multiphysics Processes

1. Fluid flow, synoptic flow, convection, turbulence
2. Radiation from the sun
3. Micro-Physics, rain formation
4. Ice growth, snow dynamics

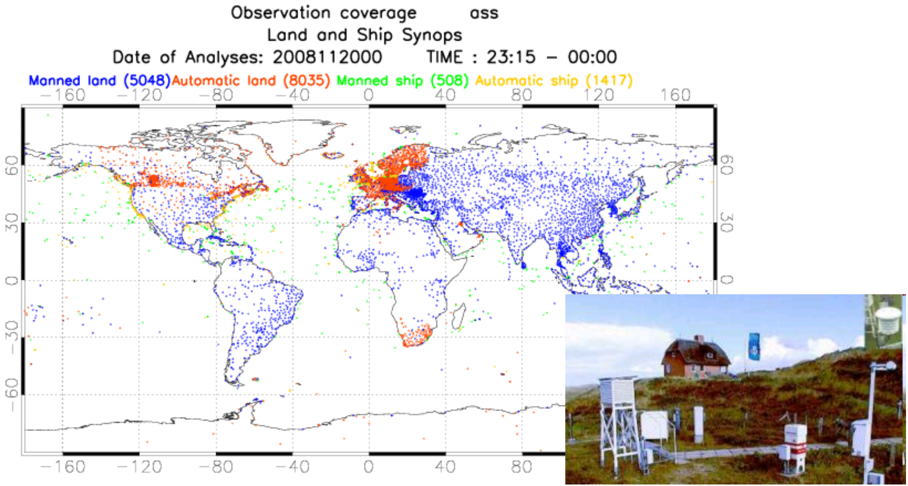
Outline

Data Survey ...

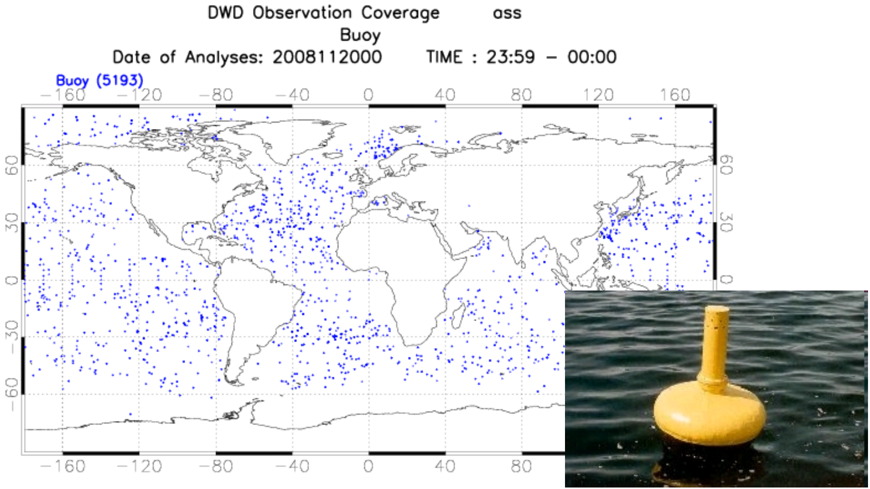


Synop, TEMP, Radiosondes, Buoys, Airplanes, Radar, Wind Profiler, Scatterometer, Radiances, GPS/GNSS, Ceilometer, Lidar

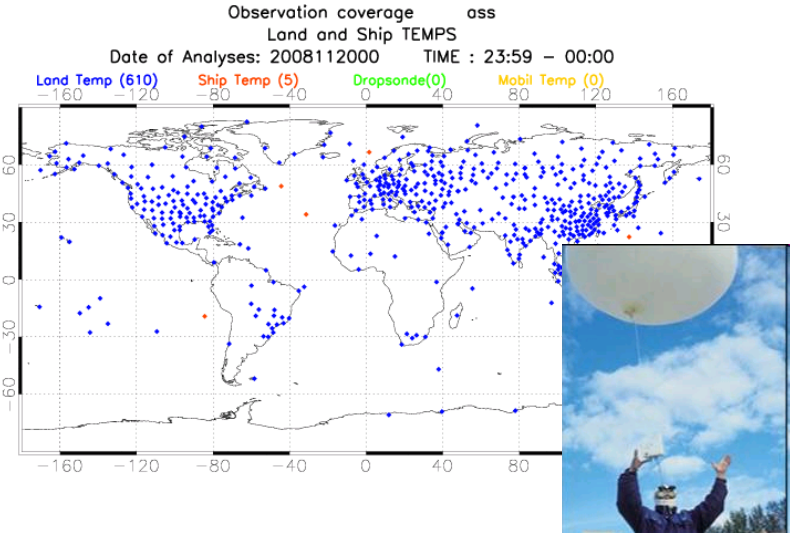
Synop ...


Observation coverage ass
Land and Ship Synops
Date of Analyses: 2008112000 TIME : 23:15 – 00:00
Manned land (5048) Automatic land (8035) Manned ship (508) Automatic ship (1417)
–160 –120 –80 –40 0 40 80 120 160
60 30 0 –30 –60

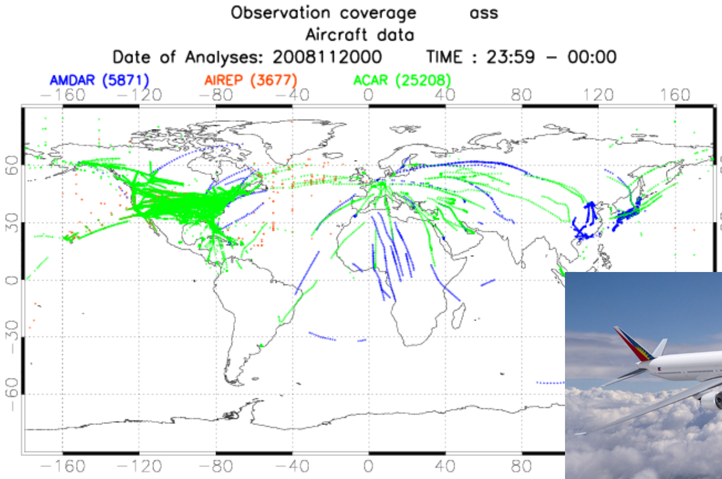
Buoys ...


DWD Observation Coverage ass
Buoy
Date of Analyses: 2008112000 TIME : 23:59 – 00:00
Buoy (5193)
–160 –120 –80 –40 0 40 80 120 160
60 30 0 –30 –60

Radio-Sondes ...

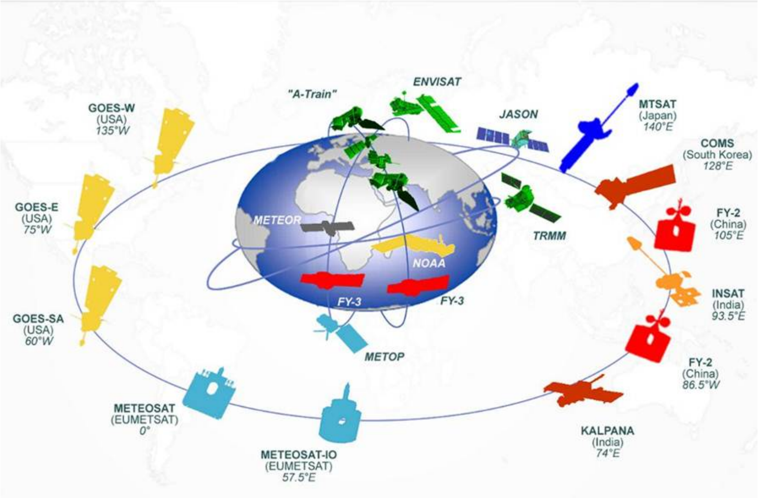


Aircrafts ...

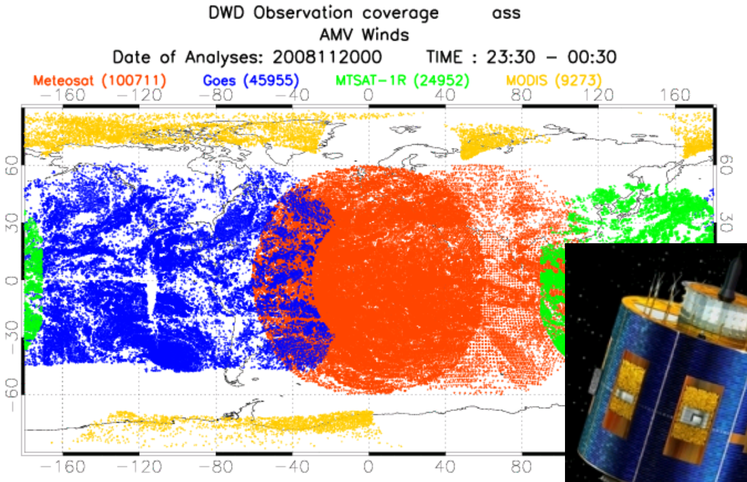

Observation coverage ass
Aircraft data
Date of Analyses: 2008112000 TIME : 23:59 – 00:00
AMDAR (5871)
AIREP (3677)
ACAR (25208)
−160 −120 −80 −40 0 40 80 120 160
60 30 0 −30 −60

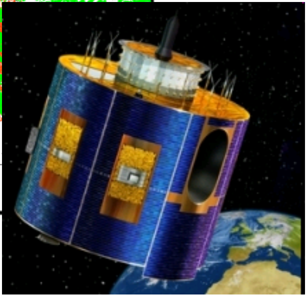

Philippines

Satellites ...

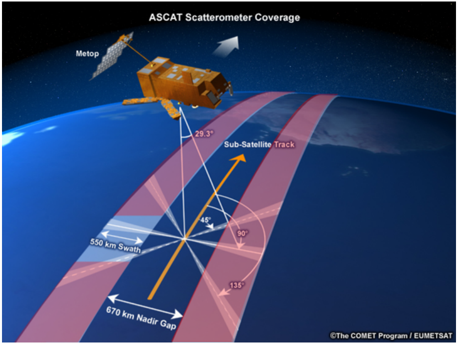

"A-Train"
ENVISAT
JASON
MTSAT
(Japan)
140°E
COMS
(South Korea)
128°E
FY-2
(China)
105°E
INSAT
(India)
93.5°E
FY-2
(China)
86.5°W
KALPANA
(India)
74°E
METEOSAT-IO
(EUMETSAT)
57.5°E
METEOSAT
(EUMETSAT)
0°
GOES-SA
(USA)
60°W
GOES-E
(USA)
75°W
GOES-W
(USA)
135°W
METEOR
NOAA
TRMM
FY-3
FY-3
METOP

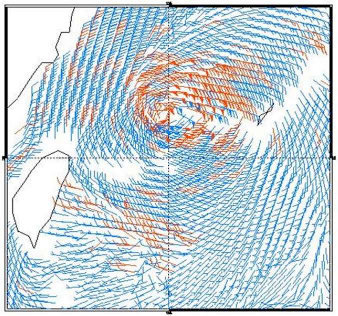
AMV Winds ...


DWD Observation coverage ass
AMV Winds
Date of Analyses: 2008112000 TIME : 23:30 – 00:30
Meteosat (100711)
Goes (45955)
MTSAT–1R (24952)
MODIS (9273)
–160 –120 –80 –40 0 40 80 120 160
60 30 0 –30 –60
–160 –120 –80 –40 0 40 80

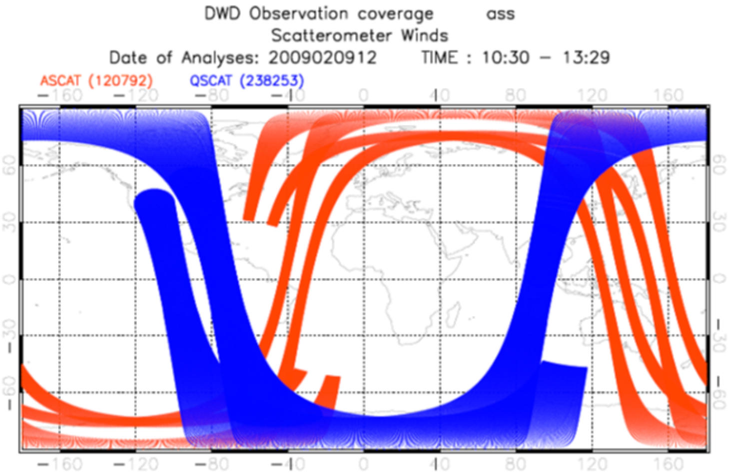


Scatterometer Winds 1 ...

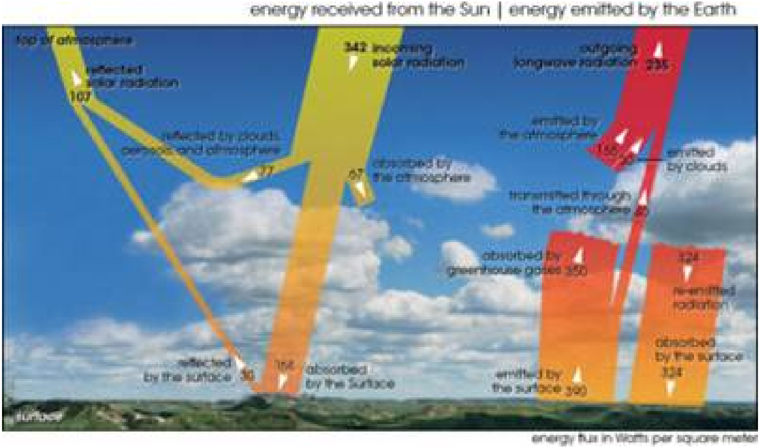




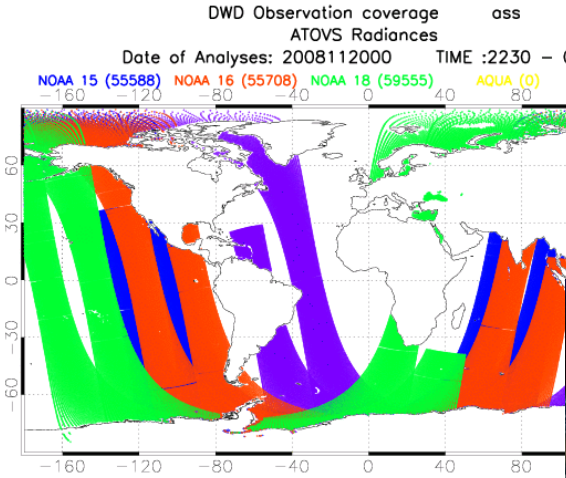
Scatterometer Winds 2 ...

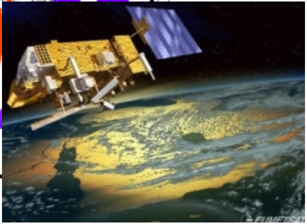


Radiances 1 ...

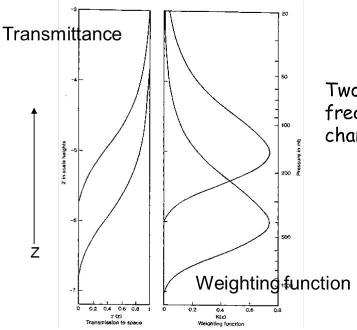

energy received from the Sun | energy emitted by the Earth
top of atmosphere
reflected solar radiation
107
342 incoming solar radiation
outgoing longwave radiation 235
reflected by clouds, aerosol and atmosphere
77
67 absorbed by the atmosphere
emitted by the atmosphere
165
30
emitted by clouds
transmitted through the atmosphere 40
absorbed by greenhouse gases 350
324
re-emitted radiation
absorbed by the surface
324
reflected by the surface 30
168 absorbed by the Surface
emitted by the surface 390
surface
energy flux in Watts per square meter

Radiances 2 ...

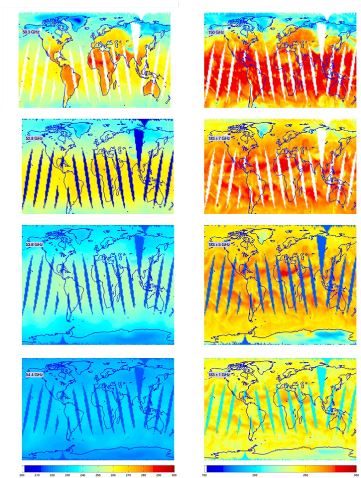

DWD Observation coverage ass
ATOVS Radiances
Date of Analyses: 2008112000 TIME :2230 – 0130
NOAA 15 (55588) NOAA 16 (55708) NOAA 18 (59555) AQUA (0) METOP (56461)
−160 −120 −80 −40 0 40 80 120 160
60 30 0 −30 −60



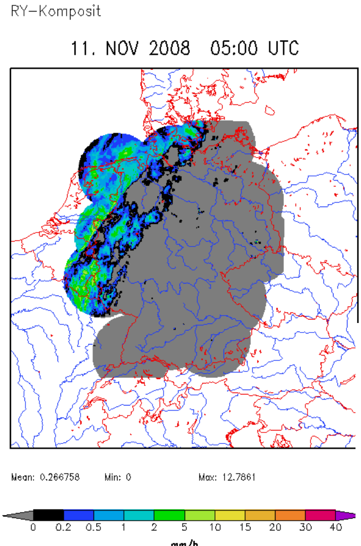
Radiances 3 ...

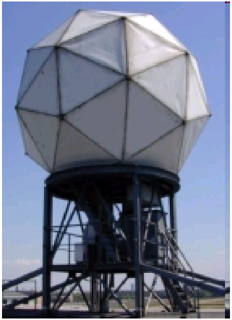


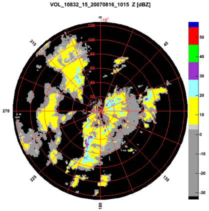
Two sounding frequencies/ channels

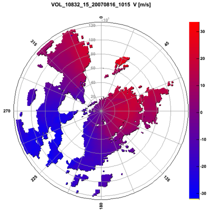


Radar ...

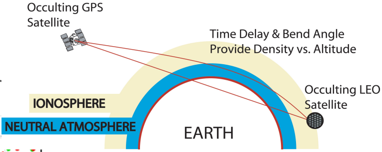

RY–Komposit
11. NOV 2008 05:00 UTC
Mean: 0.266758
Min: 0
Max: 12.7861
0 0.2 0.5 1 2 5 10 15 20 30 40
mm/h

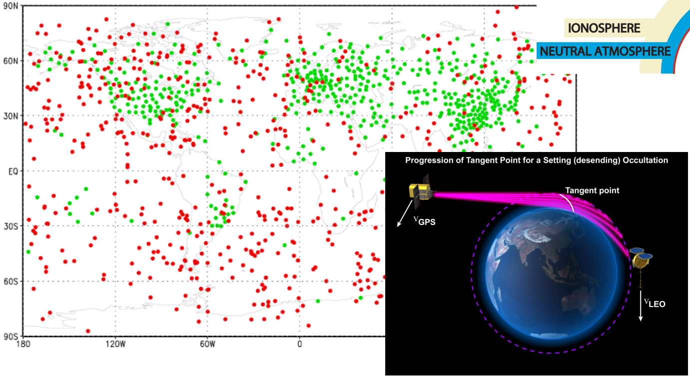



VOL_10832_15_20070816_1015 Z [dBZ]


VOL_10832_15_20070816_1015 V [m/s]

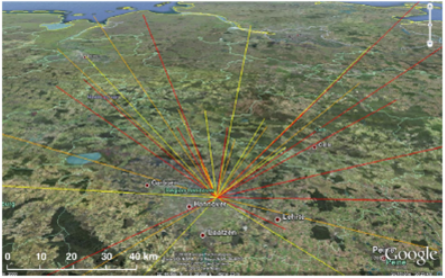
Radiooccultations ...

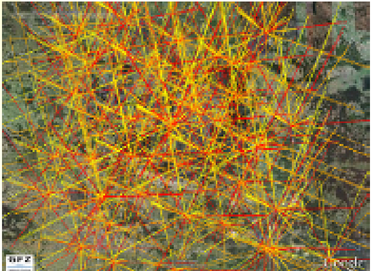




GPS Tomography ...

GPS stations operationally processed by the GFZ.





Outline

Outline

Basic Approach

Let H be the operator mapping the state x onto the measurements f. Then we need to find x by solving the equation

$$Hx = f \tag{1}$$

- Usually, the size of x is much larger than the size of f!
- Usually, H involves *remote sensing* operators!
- There is measurement error as well as numerical approximation error and model error!

When we have some initial guess x_0, we transform the equation into

$$H(x - x_0) = f - H(x_0) \tag{2}$$

and update

$$x = x_0 + H^{-1}(f - H(x_0)). \tag{3}$$

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$$H(x - x_0) = f - H(x_0) \quad (2)$$

and update

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and update

$$x = x_0 + H^{-1}(f - H(x_0)). \tag{3}$$

Regularization 1

Consider an equation

$$Hx = f \tag{4}$$

where H^{-1} is unstable or unbounded.

$$\begin{aligned} & Hx = f \\ \Rightarrow \quad & H^* Hx = H^* f \\ \Rightarrow \quad & (\alpha I + H^* H)x = H^* f. \end{aligned} \tag{5}$$

Tikhonov Regularization: Replace H^{-1} by the stable version

$$R_\alpha := (\alpha I + H^* H)^{-1} H^* \tag{6}$$

with regularization parameter $\alpha > 0$.

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Consider an equation

$$Hx = f \tag{4}$$

where H^{-1} is unstable or unbounded.

$$\begin{aligned} & Hx = f \\ \Rightarrow \quad & H^* Hx = H^* f \\ \Rightarrow \quad & (\alpha I + H^* H)x = H^* f. \end{aligned} \tag{5}$$

Tikhonov Regularization: Replace H^{-1} by the stable version

$$R_\alpha := (\alpha I + H^* H)^{-1} H^* \tag{6}$$

with regularization parameter $\alpha > 0$.

Regularization 2: Least Squares

Tikhonov regularization is equivalent to the minimization of

$$J(x) := \left(\alpha \|x\|^2 + \|Hx - f\|^2 \right) \tag{7}$$

The **normal equations** are obtained from *first order optimality conditions*

$$\nabla_x J = \frac{dJ(x)}{dx} \stackrel{!}{=} 0. \tag{8}$$

Differentiation leads to

$$\begin{aligned} & 0 = 2\alpha x + 2H^*(Hx - f) \\ \Rightarrow \quad & 0 = (\alpha I + H^* H)x - H^* f, \end{aligned} \tag{9}$$

which is our well-known *Tikhonov equation*

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Covariances and Weighted Norms

Usually, the relation between variables at different points is incorporated by using **covariances / weighted norms**:

$$J(x) := \left(\|x - x_0\|^2_{B^{-1}} + \|Hx - f\|^2_{R^{-1}} \right) \tag{10}$$

The **update formula** is now

$$\begin{aligned} x &= x_0 + (B^{-1} + H^* R^{-1} H)^{-1} H^* R^{-1} (f - H(x_0)) \\ &= x_0 + BH^*(R + HBH^*)^{-1}(f - Hx_0). \end{aligned} \tag{11}$$

Regularization 3: Spectral Methods

A singular system of an operator $W : X \to Y$ written as

$$(\mu_n, \varphi_n, g_n) \tag{12}$$

is a a set of singular values μ_n and a pair of orthonormal basis functions φ_n, g_n such that

$$\begin{aligned} H\varphi_n &= \mu_n g_n \\ H^* g_n &= \mu_n \varphi_n. \end{aligned} \tag{13}$$

We have

$$x = \sum_{n=1}^{\infty} \alpha_n \varphi_n \tag{14}$$

and

$$Hx = \sum_{n=1}^{\infty} \mu_n \alpha_n g_n. \tag{15}$$

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$$\alpha I \varphi_n = \alpha \varphi_n$$

thus

$$(\alpha I + H^* H)\varphi_n = (\alpha + \mu_n^2)\varphi_n, \quad n \in \mathbb{N}. \tag{16}$$

Consider

$$f = \sum_{n=1}^{\infty} \beta_n g_n \quad \in Y. \tag{17}$$

Tikhonov regularization $(\alpha I + H^* H)x = H^* y$ is equivalent to the spectral damping scheme

$$\alpha_n = \frac{\mu_n}{\alpha + \mu_n^2} \beta_n, \quad n \in \mathbb{N}. \tag{18}$$

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Regularization 3: Spectral Methods

True Inverse

$$x_n^{true} = \frac{1}{\mu_n} \beta_n^{true}. \tag{19}$$

This inversion is **unstable**, if $\mu_n \to 0$, $n \to \infty$!

Tikhonov Inverse (stable if $\alpha > 0$)

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Outline

Use the system dynamics!

So far we have not used the system $M : x_0 \mapsto x(t)$.

Consider some regular grid in time:

$$t_k = \frac{k}{n} T, \qquad x_k := x(t_k) = M(t_k)x_0, \quad k = 0, ..., n. \tag{21}$$

The **4dVar functional** is given by:

$$J(x) := \|x - x_0\|^2 + \sum_{k=1}^{n} \|Hx_k - f_k\|^2 \tag{22}$$

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Kalman Filter Deterministic Version

Consider the case $n = 2$. We need to minimize

$$\left\|x - x_0\right\|^2_{B^{-1}} + \left\|HM_0 x - f_1\right\|^2 + \left\|HM_1 x - f_2\right\|^2 \tag{23}$$

Decompose it into

$$J_1(x) = \left\|x - x_0\right\|^2_{B^{-1}} + \left\|HM_0 x - f_1\right\|^2 \tag{24}$$

and

$$J_2(x) = \left\|x - x_1\right\|^2_{\tilde{B}^{-1}} + \left\|HM_1 x - f_2\right\|^2 \tag{25}$$

where $\tilde{B}^{-1}$ is chosen such that

$$\left\|x - x_1\right\|^2_{\tilde{B}^{-1}} = \left\|x - x_0\right\|^2_{B^{-1}} + \left\|HM_0 x - f_1\right\|^2 + c \tag{26}$$

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Kalman Update Formula for the weights (with R error covariance matrix)

$$B_{k+1}^{-1} = B_k^{-1} + M_k^* H^* R^{-1} H M_k, \quad k = 1, 2, \ldots \tag{27}$$

and for the mean

$$x_{k+1} = x_k + B_k M_k^* H^* \left(R + H M_k B_k^{(b)} M_k^* H^*\right)^{-1} \left(f_{k+1} - H M_k x_k\right), \quad k = 1, 2, \ldots \tag{28}$$

Theorem

For linear systems and linear observation operators 4dVar and the Kalman Filter are equivalent.

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Regularization 4: Bayesian Methods

Conditional probability

$$P(A|B) := \frac{P(A \cap B)}{P(B)}, \tag{29}$$

for A, B sets in a probability space.

Conditional probability density

$$p(x|y) := \frac{p(x,y)}{p(y)}, \quad (x,y) \in X \times Y. \tag{30}$$

From

$$p(x,y) = p(x|y) \cdot p(y) = p(y|x) \cdot p(x)$$

we obtain **Bayes' formula**

$$p(x|y) = \frac{p(x)p(y|x)}{p(y)}, \quad x \in X, \quad y \in Y. \tag{31}$$

Here $p(y)$ can be considered as a normalization constant!

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Bayes' Formula

y measurement,

x unknown state of system

$$\underbrace{p(x|y)}_{\text{posteriorprob.}} = \underbrace{\frac{1}{p(y)}}_{\textit{normalization}} \underbrace{p(x)}_{\text{priorprob.}} \underbrace{p(y|x)}_{\text{measurementprob.}}$$

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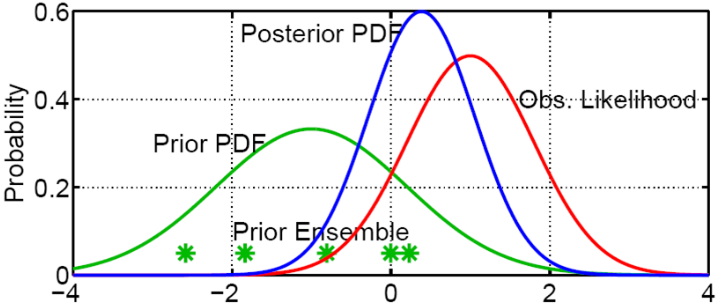
Example of Bayes



Regularization 4: Bayesian Methods

Gaussian case

$$p(x) = e^{-\frac{1}{2}x^T B^{-1} x}, \quad x \in \mathbb{R}^n$$

with **prior covariance matrix** B,

$$p(y|x) = e^{-\frac{1}{2}(y-Wx)^T R^{-1}(y-Wx)}, \quad y \in Y$$

with **measurement covariance matrix** R,

leads to the **posterior density**

$$p(x|y) = const \cdot e^{-\frac{1}{2}\left(x^T B^{-1} x + (y-Wx)^T R^{-1}(y-Wx)\right)}$$

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Regularization 4: Bayesian Methods

Maximum Likelyhood Estimator (ML)

ML: "Find the value $x \in X$ for which $p(x|y)$ is maximal"

Maximizing

$$e^{-\frac{1}{2}\left(x^T B^{-1} x + (y - Wx)^T R^{-1} (y - Wx)\right)}$$

is equivalent to **minimizing**

$$J(x) = x^T B^{-1} x + (y - Wx)^T R^{-1} (y - Wx)$$

which for $B = \alpha I$ and $R = I$ is given by

$$J(x) = \alpha ||x||^2 + ||Wx - y||^2.$$

The minimum ist calculated by the **Tikhonov operator**.

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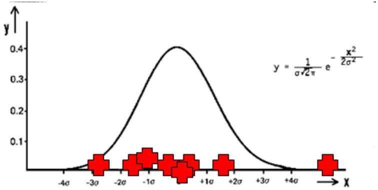
Outline

Use an ensemble of states



- Employ an ensemble of states to capture the distribution of possibilities!
- Use stochastic estimators to dynamically calculate the variances and covariances of the distribution.

Kalman Update Formula

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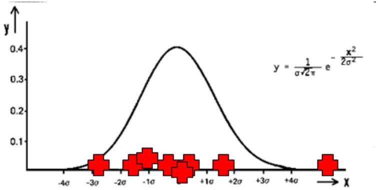
with $B_k^{(b)}$ via stochastic estimator
and for the mean

$$x_k^{(a)} = x_k^{(b)} + B_k^{(b)} H^* (R + H B_k^{(b)} H^*)^{-1} (f_k - H x_k^{(b)})$$

- $\Longrightarrow$ very efficient way to calculate the update of the weight matrix
- But does calculations only in a low dimensional subspace! Poor approximation?!

Use an ensemble of states



$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

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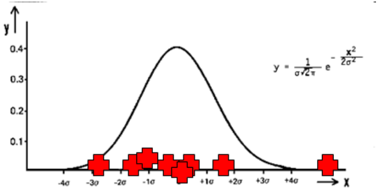

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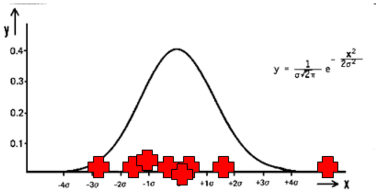

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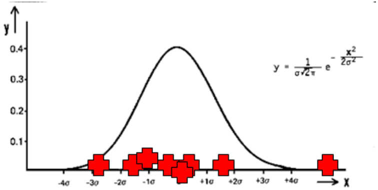

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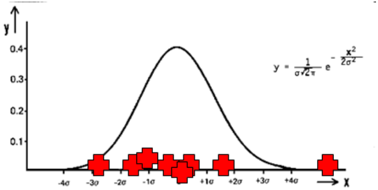

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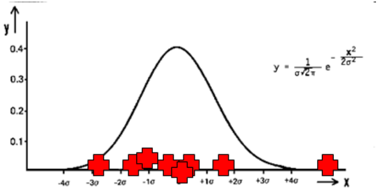

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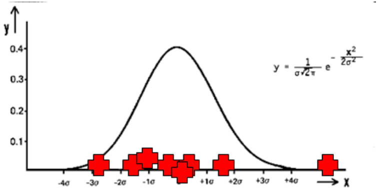

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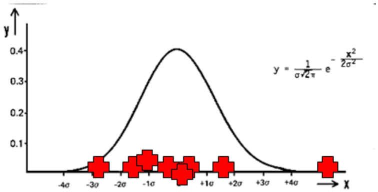

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Outline

LETKF Basic Idea

- Transform the states: work in the **ensemble space**!
- **Localize** all calculations!

Kalman Update Formula for the weights (with R error covariance matrix)

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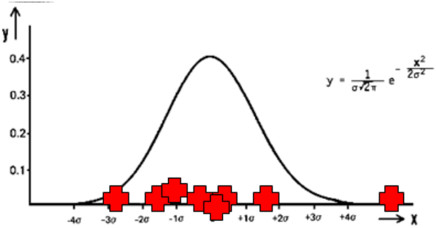

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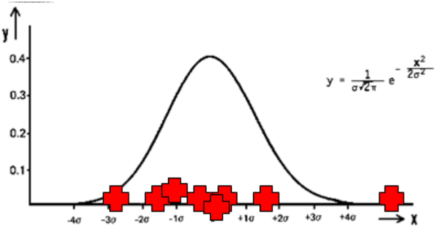

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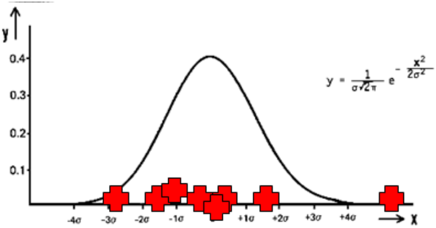

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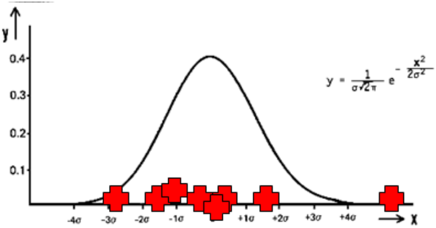

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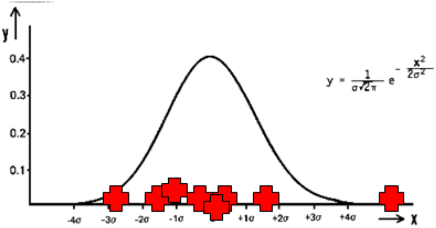

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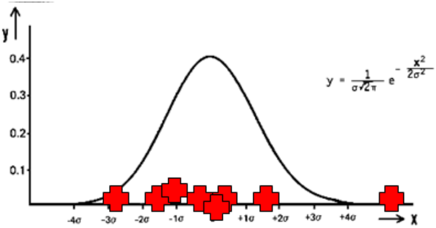

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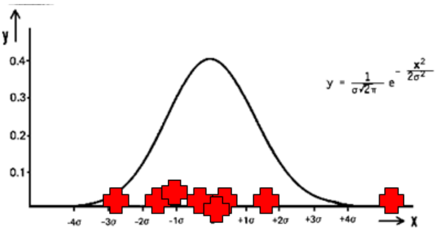

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3. Stability and instability for cycled problems
4. Localization and convergence
5. Localization for practical problems: tomographic data?!
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