Uniqueness of Current Reconstructions for Magnetic Tomography in Multi-Layer Devices

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Abstract

Magnetic tomography is an important emerging technique for the nondestructive investigation and monitoring of electrical devices. Measurements of the magnetic field of the currents in a device are used to reconstruct the current distribution. Here we investigate the uniqueness problem for current reconstructions from *multi-layer devices*. The general magnetic tomography problem is well-known to be highly non-unique and unstable. Here, as a new result for the single- and multi-layer device case we will obtain *full uniqueness* for the current reconstructions. We will base our uniqueness proof on the uniqueness of wave source splitting combined with tools from potential theory and explicit estimates for particular surface integrals involving the Biot-Savart integral operator.

1 Introduction

Fuel cells are chemical devices which transform chemical energy into electrical energy. The basic principle for a PEM (proton exchange membrane) fuel cell is shown in Figure 1. At the anode (-) hydrogen is inserted. Air or oxygen, respectively, is fueled at the cathode (+). They are separated by a semi-permeable membrane for protons coated with some catalysor (for example platinum). Protons move to the cathode through the membrane. This creates a potential which then drives electrones through an external wire and power some motor or light. Hydrogen and oxygen react at the cathode to form water and heat. A survey on reaction principles, fuel cell types and application in the portable, mobile and stationary segment can be found in [3].

Usually, the current density distribution in a fuel cell is not homogeneous over the whole active area of the cell. The supply of reactands, the contact pressure or the structure of the flowfields have great influence on the homogeneity of the current density distribution, so both for reasearch and development and for maintenance the knowledge of the current density distribution is of importance. In recent years several methods have been developed to monitor the current density distribution in fuel cells or similar layered devices. [17] provides a comparison of three in-situ methods for the determination of this distribution, in particular the *partial MEA approach*, the analysis of the currents via isolated so-called *subcells* and *current distribution mapping* via some passive resistor network integrated into the MEA. [18] presents a segmented cell method, where the voltage drop over each segment is not measured via resistors but via hall sensors. All these methods need to assemble the measurement devices into the cell. In [7] an invasive segmented cell method is combined with the noninvasive neutron radiographic imaging to study the effects of local water content on the local performance of the cell.

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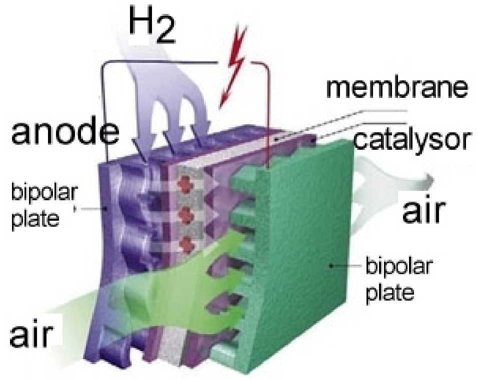

Figure 1: We show the principle of a PEM fuel cell. Hydrogen and oxygen are fueled into different layers. They react, creating a potential which drives electric currents through the wires.

Magnetic tomography is an important emerging technique where current distributions are investigated via measurements of magnetic fields. Magnetic tomography is a *nondestructive* and *noninvasive* technique, i.e. we do not need to build measurement technology into devices. This is of high importance, since the assembly of measurement technology into devices significantly alters the physical device. This leads to modified setups and is not practical for industrial production, quality control and monitoring of active systems. For a detailed description of the method we refer to [9] and [13]. In [14] the resolution of the reconstructed current density depending on the relative error is discussed. But magnetic tomography is not only limited to the fuel cell application: it is also used in biomedical imaging (see [16], [10]) to detect sources of electric currents in the human body, for example in the brain or in the muscels.

Magnetic tomography in its general setup is known to be highly non-unique, compare [12], [8]. There is a large variety of current distributions j in some domain Ω which generate the same magnetic field H in the open exterior $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$ of Ω. However, for special situations we obtain uniqueness for current reconstructions. For example, when the current is flowing in a wire grid, the unique reconstructability is shown in [8].

The Biot-Savart law is a classical tool, but still of interest in current research (see e.g. [4], [10]). Let $j \in (L^2(\Omega))^3$ be a current density and W the Biot-Savart operator (compare [13]), which maps the current density onto its magnetic field H, see also (3). By application of Green's theorem Hauer, Kühn and Potthast [8] show that the nullspace $N(W)$ of W is given by

$$N(W) = \big\{ \text{curl } v : \ v \in H_0^1(\Omega), \ \text{div } v = 0 \big\}, \tag{1}$$

which can also be seen as a consequence of the *de Rham* theory (see for example [1]). Thus, taking any compactly supported function $v \in H^1(\Omega)$ with div $v = 0$ we obtain an element $w := \text{curl } v$ which is in the nullspace $N(W)$ of W. Moreover, the authors in [8] show that the orthogonal space $N(W)^\perp$ is given by

$$\begin{aligned} N(W)^\perp \ &:= \ \big\{ j \in H_{div=0}(\Omega) : \ \langle j, j_0 \rangle_{L^2(\Omega)} = 0 \forall j_0 \in N(W) \big\} \\ &= \ \big\{ j \in H_{div=0}(\Omega) : \ \exists q \in L^2(\Omega) \text{ s.th. grad } q = \text{curl } j \big\}. \end{aligned} \tag{2}$$

In particular, *harmonic* vectorfields (defined by curl $j = 0, \text{div } j = 0$) are a subset of $N(W)^\perp$. Via curl curl $= -\triangle + \text{grad div}$ we obtain that the components of functions in $N(W)^\perp$ solve $\triangle j = 0$ in a weak sense. Thus, in general the set of reconstructable densities is strongly limited.

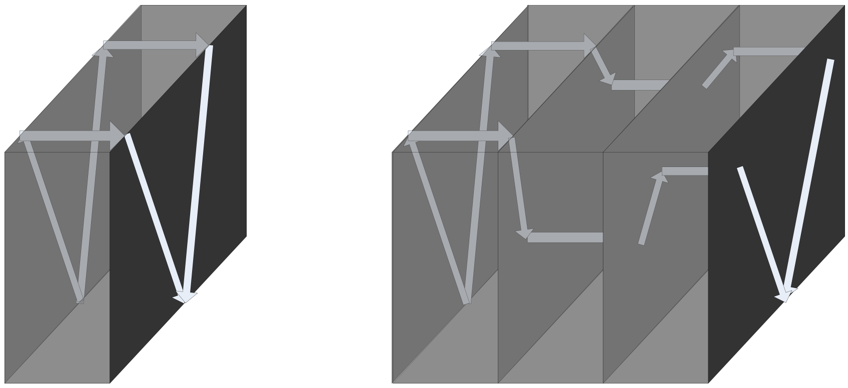

Figure 2: The figure shows the setting of a single-layer device (left) and a multi-layer device with three layers (right). The single-layer device consists of two finite plates where the current is inserted into one plate and distributes itself flowing in the plate. It can flow through the layer only in the perpendicular direction. When reaching the second plate it is collected within the plate into some point where it leaves the device. The *layer* is the space between the two plates, where the current can flow only in the direction perpendicular to the plates. The three-layer device consists of four plates and three layers between the plates.

Here we will investigate the special situation of a *single-* and *multi-layer device*. This setting is typical for magnetic tomography for fuel cells and it refers to a setup shown in Figure 2. For the single-layer case one layer is surrounded by two finite plates. Currents flow into one plate, then through the layer and are collected via the second plate. A multi-layer device consists of a collection of such coupled single-layer devices. Since the conductivity in the plates is much higher than in the layer, we assume that the currents through the layer only flow perpendicular to the end plates, which will be the key ingredient to the model studied here. This is a reasonable first step simplification, because if grid structured flow field are used (see Figure 3), the geometry limits forces the currents to flow nearly perpendicular to the membrane electrode assembly.

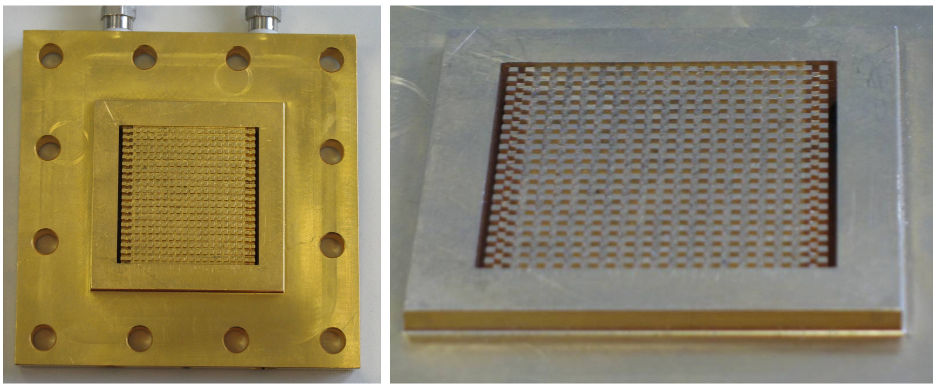

Figure 3: Grid structured flow field (left), detail view (right)

Under this condition we will prove *uniqueness* for the current reconstruction from magnetic field measurements. In particular, we investigate a measurement setting, where either the full magnetic field or its normal component is measured on a surface surrounding the device or on different surface patches in the open exterior of the set Ω. In this case as shown in [13] the measurements uniquely determine the magnetic field in the exterior Ω_e of the device Ω.

The main ingredients of our proofs are *analyticity arguments*, results from potential theory such as *uniqueness* of *exterior boundary value problems* and an analysis of the particular *local behaviour of magnetic fields* of *currents in surfaces*. The local behaviour of the magnetic field through surfaces is analysed by explicit estimates of the Biot-Savart operator in neighbourhoods of such surfaces. In principle, we expect the arguments to be relevant also to a variety of settings such as special geometries in biomedical imaging, not only for the fuel cell application.

We start in Section 2 with a detailed description of the setup of magnetic tomography in the general case and the single-layer device. Then, in Section 3 we develop a source splitting procedure which identifies magnetic fields arising from different regions in space when their superposition is measured. This is an important step towards our uniqueness proof and of interest by itself. In Section 4 we study the reconstruction of a current density in a single plane from its magnetic field. Further, we prepare results from potential theory. Finally, Section 5 collects all preparations and shows uniqueness of current reconstructions for single-layer and multi-layer devices.

2 The Setup of Magnetic Tomography

The goal of this part is the description of the setup of magnetic tomography for single- and multi-layer devices. In general, magnetic tomography is concerned with the reconstruction of a current density j defined in some bounded set $\Omega \subset \mathbb{R}^3$. Magnetic fields H of currents j are calculated via the *Biot-Savart integral operator*, defined by

$$(Wj)(x) := \frac{1}{4\pi} \int_\Omega \frac{j(y) \times (x-y)}{|x-y|^3} \, dy, \quad x \in \mathbb{R}^3 \tag{3}$$

for $j \in L^2(\Omega)$. For details about this representation and its relation to the Maxwell equations we refer to [13].

First, we will study the following *single-layer device*. We define the geometry of two end plates of the device by

$$\begin{aligned} \Gamma_1 &= \big\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : \ a_1 \le y_1 \le b_1, \ a_2 \le y_2 \le b_2, \ y_3 = c_1 \big\} \\ \Gamma_2 &= \big\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : \ a_1 \le y_1 \le b_1, \ a_2 \le y_2 \le b_2, \ y_3 = c_2 \big\}. \end{aligned} \tag{4}$$

The layer betwen the end plates is given by

$$\Lambda = \big\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : \ a_1 \le y_1 \le b_1, \ a_2 \le y_2 \le b_2, \ c_1 < y_3 < c_2 \big\} \tag{5}$$

and we define $\Omega := \Gamma_1 \cup \Gamma_2 \cup \Lambda$. In the plates Γ_1, Γ_2 we assume the current j to flow only in y_1-y_2-direction, i.e.

$$j(y) = \begin{pmatrix} j_1(y) \\ j_2(y) \\ 0 \end{pmatrix}, \quad y \in \Gamma_1 \cup \Gamma_2. \tag{6}$$

In the layer Λ we assume that the current j is flowing only in y_3-direction, i.e.

$$j(y) = \begin{pmatrix} 0 \\ 0 \\ j_3(y) \end{pmatrix}, \quad y \in \Lambda. \tag{7}$$

We like to indicate, that the currents in Γ_1 and Γ_2 are *surface currents* while the currents in Λ are *volume currents*. For our devices the currents are divergence free, i.e. we have

$$\operatorname{div} j(y) = 0, \quad y \in \Omega \tag{8}$$

and j fulfills the continuity condition

$$\nu \cdot j|_\Lambda = \operatorname{Div} j|_{\Gamma_1 \cup \Gamma_2} \tag{9}$$

with the surface divergence Div .

This leads to the conclusion that for $y \in \Lambda$ we have

$$j(y) = j(\tilde{y}) \text{ if } y_1 = \tilde{y}_1 \text{ and } y_2 = \tilde{y}_2, \tag{10}$$

i.e. the current j in Λ does depend only on the y_1, y_2 components and is constant along lines parallel to the y_3-axis.

Definition 2.1 A single-layer device *is determined by its geometry as defined by (4) - (5) with divergent-free current flow restricted by (6) - (7).*

Analogously, we define a *multi layer device*. Roughly speaking, a multi layer device consists of n single layer devices placed one after the other. In the magnetic tomography application for fuel cells this multi-layer device corresponds to a fuel cell stack with n single cells connected in series.

Definition 2.2 *A* multi-layer device *consists of* $n+1$ *plates*

$$\Gamma_k = \left\{(y_1, y_2, y_3) \in \mathbb{R}^3 : \ a_1 \le y_1 \le b_1, \ a_2 \le y_2 \le b_2, \ y_3 = c_k\right\} \tag{11}$$

for $k = 1, \dots n+1$ *with* $c_1 < c_2 < \dots < c_{n+1}$ *and* n *layers*

$$\Lambda_\ell = \left\{(y_1, y_2, y_3) \in \mathbb{R}^3 : \ a_1 \le y_1 \le b_1, \ a_2 \le y_2 \le b_2, \ c_\ell < y_3 < c_{\ell+1}\right\} \tag{12}$$

for $\ell = 1, \dots n$. *For the multi-layer device we define*

$$\Omega := \bigcup_{k=1}^{n+1} \Gamma_k \cup \bigcup_{\ell=1}^{n} \Lambda_\ell. \tag{13}$$

As in (4) and (5) we assume the currents to flow only in $y_1 - y_2$*-direction in the plates* Γ_k *and to flow only in* y_3*-direction in* Λ_ℓ*. The currents are divergence free, so we have*

$$\operatorname{div} j(y) = 0, \quad y \in \Omega, \tag{14}$$

which leads to

$$j(y) = j(\tilde{y}) \text{ if } y_1 = \tilde{y}_1 \text{ and } y_2 = \tilde{y}_2 \tag{15}$$

for $y, \tilde{y} \in \Lambda_\ell$.

Finally, we assume throughout this work that we are given measurements of the magnetic field H on some set $M \subset \Omega_e$ in the exterior of the device Ω which fully determine the magnetic field in Ω_e. This can be the normal values $\nu \times H$ on a closed surface ∂G where $G \supset \overline{\Omega}$ or the values of H on an arbitrary open set. For details about different possibilities we refer the reader to [13].

3 Source Splitting for Magnetic Fields

One of our main tools will be splitting for magnetic fields according to their region of support. Splitting of fields has been recently used in acoustic scattering by Liu and Potthast [2] in the framework of acoustic inverse scattering problems. We will see that the main ideas also apply to magnetic tomography and are an important component for the uniqueness proof below. However, the results of this section are of interest in a more general framework since they provide an analytical tool to split magnetic fields arising from different regions in space.

DEFINITION 3.1 *We say that a magnetic field H defined in $\mathbb{R}^3$ is* supported *in a domain $G \subset \mathbb{R}^3$, if there is a current distribution $j \in L^2(G)^3$ such that*

$$H = Wj \tag{16}$$

with the Biot-Savart operator W given by (3). The field H is supported on a surface Γ, if $j(x) \equiv 0$ for $x \notin \Gamma$ and (16) is satisfied, where now the volume integral is replaced by a surface integral

$$H(x) = \operatorname{curl} \int_\Gamma \Phi(x,y) j(y)\, ds(y), \quad x \in \mathbb{R}^3. \tag{17}$$

For the preparation of the following result we summarize a couple of well-known facts. Consider the representation (3), rewritten as

$$H(x) = (Wj)(x) = \operatorname{curl} \int_\Omega \Phi(x,y) j(y)\, dy. \tag{18}$$

First, from div curl $= 0$ we observe div $H = 0$ in Ω_e. Next, via curl curl $= -\triangle +$ grad div and the Maxwell equation curl $H = j = 0$ on $\Omega_e = \mathbb{R}^3 \setminus \Omega$ we obtain

$$\triangle H = \triangle H - \operatorname{grad} \operatorname{div} H = -\operatorname{curl} \operatorname{curl} H = 0 \quad \text{in } \Omega_e, \tag{19}$$

i.e., all components of a magnetic field H supported in Ω solve the Laplace equation in Ω_e.

THEOREM 3.2 (UNIQUENESS OF FIELD SPLITTING) *Consider two domains G_1, G_2 with*

$$\overline{G_1} \cap \overline{G_2} = \emptyset \tag{20}$$

and a magnetic field H supported on $G = G_1 \cup G_2$. Then the magnetic field H can be uniquely *split into the sum of two fields $H^{(1)}$ and $H^{(2)}$, respectively, which correspond to a magnetic field supported in G_1 and a field supported in G_2. Moreover, the uniqueness holds for every component H_j, $j = 1,2,3$ of a magnetic field, i.e. if H_j is supported on $G_1 \cup G_2$, then it is uniquely split into the sum of components supported on G_1 and G_2, respectively.*

Proof. Existence is clear since we assume that H is supported in $G_1 \cup G_2$ and, thus, can be represented as

$$\begin{aligned} H(x) &= \operatorname{curl} \int_{G_1} \Phi(x,y) j(y)\, dy + \operatorname{curl} \int_{G_2} \Phi(x,y) j(y)\, dy, \\ &= H^{(1)}(x) + H^{(2)}(x), \quad x \in \mathbb{R}^3. \end{aligned} \tag{21}$$

The key question is uniqueness of the splitting. Assume that there are two representations

$$H = H^{(1)} + H^{(2)} = \tilde{H}^{(1)} + \tilde{H}^{(2)}.$$

We take the difference of the two representations and define $\hat{H}^{(j)} := H^{(j)} - \tilde{H}^{(j)}$ for $j = 1, 2$. The sum $\hat{H} := \hat{H}^{(1)} + \hat{H}^{(2)}$ is zero in $\mathbb{R}^3 \setminus \overline{G}$. To guarantee sufficient smoothness up to the boundary we choose domains $\tilde{G}_j$, $j = 1, 2$ with boundary of class C^2 such that $\tilde{G}_j$ contains $\overline{G_j}$ in its interior and such that

$$\overline{\tilde{G}_1} \cap \overline{\tilde{G}_2} = \emptyset. \tag{22}$$

According to (19) the field $\hat{H}^{(1)}$ is a solution to the Laplace equation in a neighbourhood of $\tilde{G}_2$. On $\partial\tilde{G}_2$ it has the boundary values

$$\hat{H}^{(1)}|_{\partial\tilde{G}_2} = -\hat{H}^{(2)}|_{\partial\tilde{G}_2}, \quad \frac{\partial \hat{H}^{(1)}}{\partial \nu}|_{\partial\tilde{G}_2} = -\frac{\partial \hat{H}^{(2)}}{\partial \nu}|_{\partial\tilde{G}_2}. \tag{23}$$

We now define a vector field V by

$$V(x) := \begin{cases} \hat{H}^{(1)}(x) & x \in \overline{\tilde{G}_2} \\ -\hat{H}^{(2)}(x) & x \in \mathbb{R}^3 \setminus \overline{\tilde{G}_2}. \end{cases} \tag{24}$$

Then the vector field V satisfies the Laplace equation in $\tilde{G}_2$ and $\mathbb{R}^3 \setminus \overline{\tilde{G}_2}$. The field V is continuous on $\partial\tilde{G}_2$ and has continuous normal derivatives. Therefore it establishes an entire vectorial solution to the Laplace equation with the decay

$$|V(x)| \leq \frac{C}{r^2}, \quad r = |x|$$

for $|x|$ sufficiently large. From the maximum principle applied to each of its components we now obtain $V \equiv 0$. This yields $H^{(j)} = \tilde{H}^{(j)}$ for $j = 1, 2$ and the proof of the general splitting result is complete. Since in (23), (24) we have argued component wise with the Laplace equation, the uniqueness result applies to each component separately and we obtain the second statement of the theorem. □

We will employ the splitting to identify the magnetic fields coming from different plates of a single- or multi-layer device, where for simplicity of presentation here we consider the generic single-layer case. Denote the magnetic field which is generated via the Biot-Savart law (3) by the currents j in Γ_j by $H^{(j)}$ for $j = 1, 2$ and the field generated by the currents j in Λ by $H^{(3)}$. Then we have

$$H(x) = H^{(1)}(x) + H^{(2)}(x) + H^{(3)}(x), \quad x \in \mathbb{R}^3, \tag{25}$$

arising from the standard decomposition of the integration domain of (3) into

$$\Omega = \Gamma_1 \cup \Gamma_2 \cup \Lambda.$$

However, this decomposition is not adequate to obtain uniqueness results, since the domains of support do not have a positive distance. However, for the third component of the field we calculate

$$H^{(3)}(x) = \frac{1}{4\pi} \int_{\Lambda} \frac{j(y) \times (x - y)}{|x - y|^3} \, dy$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\int_\Lambda \begin{pmatrix} 0 \\ 0 \\ j_3(y) \end{pmatrix} \times \frac{x-y}{|x-y|^3}\, dy, \quad x \in \Omega_e \\
&= \frac{1}{4\pi}\int_\Lambda \frac{j_3(y)}{|x-y|^3} \begin{pmatrix} -(x_2-y_2) \\ (x_1-y_1) \\ 0 \end{pmatrix} dy, \quad x \in \Omega_e
\end{aligned} \tag{26}
$$

Thus, we obtain

$$
H_3(x) \;=\; H_3^{(1)}(x) + H_3^{(2)}(x), \quad x \in \mathbb{R}^3 \setminus (\Gamma_1 \cup \Gamma_2)\,. \tag{27}
$$

This means that the third component H_3 of the magnetic field H of a single-layer device is generated only by the currents in the end plates Γ_1 and Γ_2.

Now, we apply the splitting Theorem 3.2 to the third component H_3 of the magnetic field as given by (27) to obtain the following Corollary.

Corollary 3.3 *From the knowledge of the third component $H_3(x)$ for $x \in \Omega_e$ we can uniquely identify the the third components $H_3^{(1)}(x)$ and $H_3^{(2)}(x)$ supported on the end plates Γ_1 and Γ_2 of the single-layer device.*

4 Uniqueness of the Reconstruction of Currents in Surfaces and Further Preparations

The goal of this section is to show uniqueness for the current reconstruction when the field arises from a current density supported in a planar surface. Further, we prepare results from potential theory for later use and provide some particular uniqueness properties arising from our geometrical setup. We start with the generic situation of reconstructing a current flowing in a full straight two-dimensional plane Γ in $\mathbb{R}^3$ from the full knowledge of its magnetic field in $\mathbb{R}^3 \setminus \Gamma$.

Theorem 4.1 *Consider the magnetic field H of a piecewise continuous current density j flowing through a plane Γ in $\mathbb{R}^3$ and a point $z_0 \in \Gamma$ where j is continuous. Then we have the behaviour*

$$
\lim_{x \to z_0} H(x) \times \nu = \frac{1}{2}\, j(z_0), \tag{28}
$$

where ν denotes the normal vector for Γ.

Proof. First we mention, that this behaviour of the magnetic field corresponds to the jump relation for vector potentials (see Thm. 6.11 in [5]). For the sake of self-containedness we present the proof for this special case. We choose a coordinate system such that $z_0 = 0$ and $\Gamma = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$. Further, we rotate the $x_1 - x_2$-plane such that $j(0) = e_2|j(z_0)|$, i.e the current density in the origin points into the x_2-direction. As a *first* step consider a constant current j_0. Then, we calculate $H(x)$ in the point $x_h := (0, 0, h)$ with $h > 0$. In this case the magnetic field has components only in the x_1 and x_3 direction given by

$$
H(x) \;=\; \frac{1}{4\pi}\int_\Gamma \frac{x-y}{|x-y|^3} \times \begin{pmatrix} 0 \\ j_{0,y} \\ 0 \end{pmatrix} ds(y)
$$

$$= \frac{1}{4\pi} \int_{\Gamma} \frac{1}{|x-y|^3} \cdot \begin{pmatrix} -(x_3-y_3)j_{0,y} \\ 0 \\ (x_1-y_1)j_{0,y} \end{pmatrix} ds(y). \tag{29}$$

We first study the H_1 component, which is given by

$$H_1(x_h) = -\frac{hj_{0,y}}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{h^2+y_1^2+y_2^2}^3} \, dy_1 dy_2. \tag{30}$$

We switch over to polar coordinates, i.e. we substitute $x = r\sin\theta, y = r\cos\theta$ and calculate

$$\begin{aligned} H_1(x_h) &= -\frac{hj_{0,y}}{4\pi} \int_0^{\infty} \int_0^{2\pi} \frac{r}{\sqrt{h^2+r^2\sin^2\theta+r^2\cos^2\theta}^3} \, d\theta dr \\ &= -\frac{hj_{0,y}}{2} \int_0^{\infty} \frac{r}{\sqrt{h^2+r^2}^3} \, dr = -\frac{j_{0,y}}{2}. \end{aligned} \tag{31}$$

From symmetry arguments we further obtain $H_3(x_h) = 0$. From

$$H \times \nu = \frac{1}{2} \begin{pmatrix} -j_{0,y} \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ j_{0,y} \\ 0 \end{pmatrix} \tag{32}$$

we now derive (28) for a constant current density j.

As *second* step we consider a bounded current density j which is continuous at z_0 and employ the same coordinate system as chosen above. We use (31) for the H_1 component with $j_{0,y} := j_y(0)$ to calculate

$$H_1(x_h) = -\frac{h}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j_y(y_1,y_2)}{\sqrt{h^2+y_1^2+y_2^2}^3} \, dy_1 dy_2 \tag{33}$$

$$= -\frac{h}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j_y(y_1,y_2)-j_{0,y}}{\sqrt{h^2+y_1^2+y_2^2}^3} \, dy_1 dy_2 - \frac{h}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j_{0,y}}{\sqrt{h^2+y_1^2+y_2^2}^3} \, dy_1 dy_2 \tag{34}$$

$$= -\frac{h}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j_y(y_1,y_2)-j_{0,y}}{\sqrt{h^2+y_1^2+y_2^2}^3} \, dy_1 dy_2 - \frac{j_{0,y}}{2}. \tag{35}$$

We denote the integral in (35) by $Q(h)$. Then, with polar coordinates we split the integration into a small ball with radius $R > 0$ and the rest to obtain

$$\begin{aligned} Q(h) &= -\frac{h}{4\pi} \int_0^R \int_0^{2\pi} \frac{j_y(r\sin\theta, r\cos\theta) - j_{0,y}}{\sqrt{h^2+r^2}^3} \, r \, d\theta dr \\ &\quad -\frac{h}{4\pi} \int_R^{\infty} \int_0^{2\pi} \frac{j_y(r\sin\theta, r\cos\theta) - j_{0,y}}{\sqrt{h^2+r^2}^3} \, r \, d\theta dr \end{aligned} \tag{36}$$

and estimate with universal constants C_1 and C_2 depending on j and R but not on h:

$$|Q(h)| \leq \underbrace{C_1 \|j_y - j_{0,y}\|_{\infty, B_R(0)} \left| \int_0^R \frac{hr}{\sqrt{h^2+r^2}^3} \, dr \right|}_{=:\tau(R)} + \frac{C_2 h}{\sqrt{h^2+R^2}}. \tag{37}$$

Since j_y is continuous and bounded, for given $\epsilon > 0$ we choose $R > 0$, so that $\tau(R) < \epsilon/2$. Then we choose $h_0 > 0$ so that $C_2 \frac{h_0}{\sqrt{h_0^2+R^2}} < \epsilon/2$. Thus $|Q(h)| < \epsilon$ for all $h < h_0$. Since the choice of ϵ was arbitrary we have the convergence $Q(h) \to 0$ for $h \to 0$ and thus

$$\lim_{h\to 0} H_1(x_h) = -\frac{j_{0,y}}{2}. \tag{38}$$

With analogous argumentation we can derive that $H_j(x_h) \to 0$ for $h \to 0$ for $j = 2$ and $j = 3$. Now, the statement (28) for an arbitrary current density j which is continuous in z_0 is obtained from (32). □

Further preparations are carried out by the following result for solutions to the Laplace equation in unbounded domains. We refer the reader to [11] or [15] to further details about exterior Neumann problems for the Laplace equations, where the boundaries are usually arising from bounded domains. Our goal here is to use the result that the normal component of H on a plane Γ uniquely determines H, which follows from well-known arguments via the reflection principle. For completeness and the convenience of readers from a broader audience here we present a concise version of the proof.

Theorem 4.2 (Uniqueness of exterior Neumann problem) *Consider a half-space $U \subset \mathbb{R}^3$ and the normal vector ν pointing into the interior of U. Let $\varphi \in C^2(U) \cap C^1(\overline{U})$ be a solution to the* Laplace equation

$$\triangle\varphi = 0 \quad \text{in } U \tag{39}$$

with the Neumann boundary condition

$$\nu \cdot \text{grad } \varphi = 0 \quad \text{on } \partial U \tag{40}$$

and the decay condition

$$|\varphi(y)| = o(1), \quad r = |y| \to \infty \tag{41}$$

uniformly in all directions. Then φ is equal to zero in U.

Proof. Without loss of generality we can assume that $U = \mathbb{R}^2 \times \mathbb{R}^+$, i.e. U is the half-space with positive y_3-component. We employ the *reflection principle*, i.e. we extend the field φ into $\mathbb{R}^3$ by

$$\varphi(y) := \varphi(y'), \quad y \in \mathbb{R}^2 \times \mathbb{R}^-, \tag{42}$$

where $y' = (y_1, y_2, -y_3)$ for any point $y \in \mathbb{R}^3$. This reflected function along ∂U is *continuous* in $\mathbb{R}^3$. It clearly satisfies the Laplace equation in U and in $\mathbb{R}^3 \setminus U$. We calculate

$$\frac{\partial\varphi}{\partial y_3}(y) = -\frac{\partial\varphi}{\partial y_3}(y'), \quad y \in \mathbb{R}^3. \tag{43}$$

Thus, due to the homogeneous boundary condition (40) the potential φ has a continuous normal derivative on ∂U. From this we conclude that φ is a solution to the Laplace equation in the full space $\mathbb{R}^3$. Details of the argument can be carried out via Green's representation theorem applied to φ in a ball B. In each of the subdomains $B_1 := B \cap U$ and $B_2 := B \cap (\mathbb{R}^3 \setminus U)$ the field φ satisfies the Laplace equation and we can thus represent φ by

$$\begin{aligned}\varphi(x) \quad = \quad & \int_{\partial B_1} \left\{ \frac{\partial\varphi}{\partial\nu}(y)\Phi(x,y) - \varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y) \\ & + \int_{\partial B_2} \left\{ \frac{\partial\varphi}{\partial\nu}(y)\Phi(x,y) - \varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y)\end{aligned} \tag{44}$$

where ν here denotes the normal vector to ∂B_1 or ∂B_2 directed into the exterior of these bounded domains. For the integrals in (44) on the joint boundary $S = B \cap \partial U$ we now observe that the the first term in both integrals vanishes and the second term has the same modulus but a different sign due to the different direction of the normals. Thus, we obtain

$$\varphi(x) \quad = \quad \int_{\partial B} \left\{ \frac{\partial \varphi}{\partial \nu}(y)\Phi(x,y) - \varphi(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y), \quad x \in B, \tag{45}$$

i.e. the potential φ is smooth in B and satisfies the Laplace equation. We now have an entire solution to the Laplace equation which satisfies the decay condition (41). From the maximum principle applied to φ in a large ball $B_R(0)$ with radius R we obtain

$$|\varphi(x)| \le o(1), \quad x \in B_R(0). \tag{46}$$

Taking the limit $R \to \infty$ we obtain $\varphi \equiv 0$ and the proof is complete □

Theorem 4.3 (Uniqueness for boundary value problem) *Let H be some magnetic field generated by a current distribution j in a bounded set Ω and let U be a half-space containing $\overline{\Omega}$ in its open exterior. We denote the normal vector to the boundary ∂U of U by ν. Then, the knowledge of $\nu \cdot H$ on ∂U uniquely determines H in the exterior Ω_e of Ω.*

Proof. First, we note that H satisfies the Maxwell equations and we have curl $H = 0$ in the exterior Ω_e of Ω. Since Ω_e is simply connected, there is a magnetic potential φ_H with

$$H = \text{grad } \varphi_H \quad \text{in } \Omega_e. \tag{47}$$

It is unique up to a constant, which we will determine below. Using div grad $= \triangle$ from div $(\mu H) = 0$ we obtain the *Laplace equation*

$$\triangle \varphi_H = 0 \text{ in } \Omega_e \tag{48}$$

for φ_H. From $\nu \cdot H = g$ for some function g we obtain

$$g = \nu \cdot H = \nu \cdot \text{grad } \varphi_H = \frac{\partial \varphi_H}{\partial \nu}, \tag{49}$$

i.e. we have a *Neumann boundary condition* on ∂U. Further, since H is generated by a current distribution in Ω it is represented by the Bio-Savart law (3) and we obtain the decay

$$|H(x)| \le \frac{C}{|r|^2}, \quad |x| = r \to \infty. \tag{50}$$

It is sufficient to estimate the decay of φ outside a ball $B_R(0)$ with $\Omega \subset B_R(0)$. We now choose the constant for the magnetic potential such that φ_H is zero at $(\infty, 0, 0)$. Then, we can calculate φ_H at $x_a = (-a, 0, 0)$ by

$$\varphi(x_a) = -\int_{\infty}^{a} H_1(y)dy \tag{51}$$

such that

$$|\varphi(x_a)| \le |\int_a^{\infty} \frac{c}{r^2} dr| \le \frac{c}{a}. \tag{52}$$

To obtain the potential in other points $x \in \mathbb{R}^3 \setminus B_R(0)$ with $R = |x|$ we first integrate from $(-\infty, 0, 0)$ to $(-R, 0, 0)$, then integrate along a circle of radius R around the origin which connects $(-R, 0, 0)$ and x. We estimate

$$\begin{aligned} |\varphi(x)| &\leq \frac{c}{R} + \left|\int_0^\alpha \tau(y) \cdot H(y) d\tau(y)\right| \\ &\leq \frac{c}{R} + \frac{\alpha R}{R^2} \leq \frac{c + 2\pi}{R} \end{aligned} \tag{53}$$

with some constant c. This proves the decay estimate

$$|\varphi(x)| \leq \frac{c}{r}, \quad r = |x| \to \infty \tag{54}$$

uniformly for all directions with some constant c. Finally, we now collect all parts to obtain uniqueness for the magnetic boundary value problem. Assume that there are two magnetic fields $H, \tilde{H}$ which have the same boundary values

$$\nu \cdot H = \nu \cdot \tilde{H} \quad \text{on } \partial U. \tag{55}$$

Then the field $\hat{H} := H - \tilde{H}$ satisfies a homogeneous condition on ∂U and obeys all other conditions of the fields which were exploited above. Then the magnetic potential for $\hat{H}$ solves the Neumann boundary value problem with homogeneous boundary values, thus the potential vanishes in U and this yields $H = \tilde{H}$ in U. By analyticity of these fields in $\mathbb{R}^3 \setminus \Omega$ they coincide in $\mathbb{R}^3 \setminus \Omega$ and the proof is complete. □

LEMMA 4.4 *Consider a single-layer device as in Definition 2.1. Then the currents in* Γ_1 *uniquely determine the currents in* Λ.

Proof. Without loss of generality we assume that $c_1 = 0$ and we recall the directional constraints (6) and (7). To employ a divergence in $\mathbb{R}^3$ we use distributions to write the volume current density j in a neighbourhood of the layer Γ_1 as

$$j_1(y) = \delta(y_3)\tilde{j}_1(y_1, y_2) \tag{56}$$

$$j_2(y) = \delta(y_3)\tilde{j}_2(y_1, y_2) \tag{57}$$

where $\tilde{j}$ here is the surface current density in Γ_1. For $y \in \Lambda$ we have

$$\operatorname{div} j(y) = \frac{\partial}{\partial y_3} j_3(y) = 0, \tag{58}$$

so j_3 does only depend on y_1 and y_2 and we can rewrite j_3 as

$$j_3(y) = \Theta(y_3)\tilde{j}_3(y_1, y_2) \tag{59}$$

with the Heaviside step function Θ and a function $\tilde{j}_3(y_1, y_2)$ that only depends on y_1 and y_2. From (59) we derive

$$\frac{\partial}{\partial y_3} j_3(y) = \delta(y_3)\tilde{j}_3(y_1, y_2) \tag{60}$$

with the Dirac delta funtion and consequently $\operatorname{div} j = 0$ yields

$$\delta(y_3)\tilde{j}_3(y_1, y_2) = -\frac{\partial}{\partial y_1} j_1(y) - \frac{\partial}{\partial y_2} j_2(y) = -\delta(y_3)\frac{\partial}{\partial y_1}\tilde{j}_1(y) - \delta(y_3)\frac{\partial}{\partial y_2}\tilde{j}_2(y). \tag{61}$$

Integration over y_3 now yields

$$\tilde{j}_3(y_1, y_2) = -\frac{\partial}{\partial y_1}\tilde{j}_1(y) - \frac{\partial}{\partial y_2}\tilde{j}_2(y) \tag{62}$$

and the proof is complete. □

5 Uniqueness Results for Current Reconstruction

In this last section we collect all preparations to derive our main results. We have split the presentation into two steps, where first we present a simple version of the arguments for a *single-layer* device and then work out the general *multi-layer case.*

Theorem 5.1 (Uniqueness for Single-Layer Device) *Consider a measurement setup according to Section 2 which uniquely determines the field H in the exterior of*

$$\Omega = \Gamma_1 \cup \Gamma_2 \cup \Lambda.$$

Then, for a single-layer device the currents $j(y)$, $y \in \Omega$, are uniquely determined by the measurements of H.

Proof. We decompose the magnetic field H according to (25) supported in Γ_1, Γ_2 and Λ respectively. Our measurements of the magnetic field H by assumption determine H in Ω_e. In particular, we can choose a plane $\Gamma \subset \Omega_e$ perpendicular to the x_3-axis on which H and thus also $\nu \cdot H = H_3$ is determined from our measurements.

As a first step by the use of Corollary 3.3 we uniquely identify the third components $H_3^{(j)}$ which are supported on the end plates Γ_j for $j = 1, 2$. Next, we apply Theorem 4.3 to the field $H^{(1)}$ and $H^{(2)}$ separately, stating that the third components of the fields on Γ determine the full fields. As a result, the two fields $H^{(1)}$ and $H^{(2)}$ which are supported on Γ_1 or Γ_2, respectively, are uniquely determined.

Now, we consider the surface patch Γ_1 imbedded into an infinite plane Γ in $\mathbb{R}^3$ and apply Theorem 4.1 to reconstruct the currents $j|_{\Gamma_1}$ from $H^{(1)}$. Analogously, we reconstruct the current $j|_{\Gamma_2}$ from $H^{(2)}$, i.e. the current densities in the two end plates are determined uniquely from the data.

Finally, with the knowledge of the currents in the end plates Γ_1 and Γ_2 the currents in Λ are determined by Lemma 4.4. □

Finally, we extend our results to the multi-layer device. Consider a multi-layer device as in Definition 2.2 with n layers. Then we can decompose the magnetic field according to the region of support similar as in (25). Here, our notation first labels the different plates in $x_1 - x_2$-direction and then counts the areas between these plates.

Denote the magnetic field which is generated via the Biot-Savart law by the currents j in Γ_k by $H^{(k)}$ for $k = 1, \ldots, n+1$ and the field generated by the currents j in Λ_ℓ by $H^{(n+1+\ell)}$ for $\ell = 1, \ldots n$. Then, we have

$$H(x) = \sum_{k=1}^{n+1} H^{(k)}(x) + \sum_{\ell=1}^{n} H^{(n+1+\ell)}(x), \quad x \in \mathbb{R}^3 \tag{63}$$

$$H_3(x) = \sum_{k=1}^{n+1} H_3^{(k)}(x), \quad x \in \mathbb{R}^3. \tag{64}$$

This means, the third component H_3 of the magnetic field H of a multi-layer device is generated only by the currents in the plates Γ_i, $i = 1, \dots n+1$.

THEOREM 5.2 (UNIQUENESS FOR MULTI-LAYER DEVICE) *Consider a measurement setup according to Section 2 which uniquely determines the field H in the exterior of Ω given by (13). Then, for a multi-layer device the currents $j(y)$, $y \in \Omega$, are uniquely determined by the measurements of the magnetic field H.*

Proof. We need to carefully check which arguments from the single-layer case can be carried over to the multi-layer setting. According to (63) we decompose the magnetic field H into

$$H(x) = \sum_{\xi=1}^{2n+1} H^{(\xi)}(x), \quad x \in \mathbb{R}^3 \tag{65}$$

supported in Γ_k and Λ_ℓ for $k = 1, ..., n+1$ and $\ell = 1, ..., n$ respectively. Our measurements of the magnetic field H by assumption determine H in Ω_e. Again, we choose a plane $\Gamma \subset \Omega_e$ perpendicular to the x_3-axis on which H and thus also $\nu \cdot H = H_3$ is determined from our measurements.

As a first step by the use of a generalization of Corollary 3.3 we uniquely identify the third components $H_3^{(k)}$ which are supported on the end plates Γ_k for $k = 1, ..., n+1$. To this end we remark that the arguments of source splitting as worked out in Theorem 3.2 directly generalize to multiple domains as follows. Consider a splitting into n domains G_1, .., G_n. Then we first apply Theorem 3.2 to G_1 and $G := G_2 \cup ... \cup G_n$ to obtain $H_3 = H_3^{(1)} + \mathbf{H}_3^{(2)}$. We then apply the theorem to the field $\mathbf{H}_3^{(2)}$ and split it further. After n steps we obtain a splitting of H_3 into $n+1$ uniquely determined components which are supported on Γ_k, $k = 1, .., n+1$.

After this step, we apply Theorem 4.3 to the field $H^{(k)}$ for each $k = 1, ..., n+1$ separately. We obtain that the third components of the fields on Γ_k determine the full fields. As a result, the fields $H^{(k)}$ which are supported on Γ_k are uniquely determined for all layers $k = 1, ..., n+1$ of the multi-layer device.

Now, for each $k = 1, ..., n+1$ we consider the surface patch Γ_k imbedded into an infinite plane Γ in $\mathbb{R}^3$ and apply Theorem 4.1 to reconstruct the currents $j|_{\Gamma_k}$ from $H^{(k)}$. As a consequence the current densities in each plate Γ_k is determined uniquely from the data.

Finally, with the knowledge of the currents in the plates Γ_k the currents in Λ_ℓ for $\ell = 1, ..., n$ are determined by Lemma 4.4. This completes the proof for the general case. □

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