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On the convergence of the finite integration technique for the anisotropic boundary value problem of magnetic tomography

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SUMMARY

The reconstruction of a current distribution from measurements of the magnetic field is an important problem of current research in inverse problems. Here, we study an appropriate solution to the forward problem, i.e. the calculation of a current distribution given some resistance or conductivity distribution, respectively, and prescribed boundary currents. We briefly describe the well-known solution of the continuous problem, then employ the *finite integration technique* as developed by Weiland *et al.* since 1977 for the solution of the problem. Since this method can be physically realized it offers the possibility to develop special tests in the area of inverse problems. Our main point is to provide a new and rigorous study of convergence for the boundary value problem under consideration. In particular, we will show how the arguments which are used in the proof of the continuous case can be carried over to study the finite-dimensional numerical scheme. Finally, we will describe a program package which has been developed for the numerical implementation of the scheme using Matlab. Copyright © 2003 John Wiley & Sons, Ltd.

1. INTRODUCTION

The reconstruction of current distributions is a basic task for many applications from such diverse areas as medical diagnosis to non-destructive testing and exploration, see for example [1–9]. Electric currents arise from voltages on the basis of mostly inhomogeneous resistance distributions or conductivity tensor, respectively, or they are produced by chemical processes within various industrial applications. Here, we will study the solution of the *direct* problem, i.e. the calculation of a current distribution j which satisfies the *static Maxwell equations* given an anisotropic conductivity distribution σ on some domain Ω and prescribed boundary currents

$$v \cdot j = g \quad \text{on } \partial\Omega \tag{1}$$

We consider a current flowing through a three-dimensional domain Ω, especially a cuboid with axis parallel to the co-ordinate axis. This setting is of importance for several industrial

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applications [5]. If we know the normal components of the current on the boundary $\partial\Omega$ of the domain Ω and as (in general) *anisotropic* conductivity distribution σ, respectively, the current is uniquely determined in the interior of the domain Ω. Since the results are scattered in the literature, we will summarize the basic background and sketch the derivations in Section 2.1. Currents, voltages and conductivities are connected by Ohm's law.

Then we investigate the grid model due to the *finite integration technique* developed by Weiland since 1977, see References [10,11] for a survey on this approach. The grid model of the method is chosen such that it can be *realized physically* by wires and resistance elements. Thus, we can use this grid model to test the measurement devices and the algorithm for *real-data reconstructions* for the *inverse* problem. Since for the grid model the reconstruction algorithm for the inverse problem will turn out to be *less ill-posed* than the full continuous reconstruction (see Reference [3]), this is an important *intermediate step* for the solution of the inverse problem. This importance substantiates the need for some rigorous study of the relation between the continuous and the discrete model.

We prove *solvability* of the grid model and *convergence* of the solution towards the solution of the continuous problem. In particular, we show how the arguments of the continuous model can be discretized and used to establish the convergence properties, an approach which we consider as a new and original contribution to the finite integration technique in its relation to the well-established methods for continuous boundary value problems. Also, we present some three-dimensional numerical examples for the calculation of the magnetic fields.

2. CALCULATION AND PROPERTIES OF CURRENT DENSITIES AND MAGNETIC FIELDS

The goal of this section is to collect properties of the solution of the continuous problem. Given a conductivity distribution the currents arise as a solution to an elliptic boundary value problem and the magnetic fields are given by the Biot–Savart law.

2.1. Background of the direct and inverse problems

To derive the grid model and prove convergence, we will first review the derivation of the continuous model from the Maxwell equations and briefly summarize the proof of its solvability using the Lax–Milgram Theorem.

In general, the behaviour of time-independent currents, electric and magnetic fields is governed by the stationary (or reduced) *Maxwell equations*

$$\begin{aligned} \nabla \times H = j, \quad \nabla \times E = 0 \\ \nabla \cdot D = \rho, \quad \nabla \cdot B = 0 \end{aligned} \tag{2}$$

They are complemented by the *material equations*

$$D = \varepsilon\varepsilon_0 E, \quad B = \mu\mu_0 H, \quad j = \sigma E \tag{3}$$

Here, E is the electric field, D the electric flux, H the magnetic field, B the magnetic flux, j a current distribution, ρ the current density, σ the conductivity distribution, ε the electric permittivity and μ the permeability of the medium under consideration, ε_0 and μ_0 are the well-known natural constants for the vacuum.

Let Ω be a bounded domain in $\mathbb{R}^3$ with piecewise C^2-boundary satisfying interior and exterior cone-conditions as defined in Reference [12]. The magnetic field of a current distribution $j \in L^2(\Omega)$ defined on Ω is given by the *Biot–Savart law*

$$H(x) = \frac{1}{4\pi} \int_\Omega \frac{j(y) \times (x-y)}{|x-y|^3} \, \mathrm{d}y, \quad x \in \mathbb{R}^3 \tag{4}$$

With the help of the *fundamental solution*

$$\Phi(x, y) := \frac{1}{4\pi|x-y|}, \quad x \neq y \in \mathbb{R}^3 \tag{5}$$

the field H is given by the operator

$$(Wj)(x) := \nabla_x \times \int_\Omega \Phi(x, y) j(y) \, \mathrm{d}y, \quad x \in \mathbb{R}^3 \tag{6}$$

We assume that a conductivity distribution is given in Ω and that a current distribution j is known on the boundary $\partial\Omega$ of Ω. For $j|_{\partial\Omega}$ the condition

$$\int_{\partial\Omega} v(x) \cdot j(x) \, \mathrm{d}s(x) = 0 \tag{7}$$

will be satisfied, which is a consequence of $\nabla \cdot j = 0$ in Ω. Also, we assume that no free charges are present in the system.

Please note that for a closed system of currents with $j \equiv 0$ in the exterior of some domain $\tilde{\Omega}$ the Biot–Savart law can be derived from the Maxwell equations (2). By an application of the rotation $\nabla \times$ to $\nabla \times H = j$ using $\nabla \cdot H = 0$ and $\nabla \times \nabla \times a = -\triangle a + \nabla(\nabla \cdot a)$ we obtain

$$\nabla \times j = \nabla \times \nabla \times H = -\triangle H \tag{8}$$

We solve the Poisson equation (8) by a volume potential (see Reference [13, Theorem 8.1] for the case $k = 0$), i.e. we have

$$H(x) = \int_{\tilde{\Omega}} \Phi(x, y) \nabla_y \times j(y) \, \mathrm{d}y, \quad x \in \mathbb{R}^3 \tag{9}$$

A partial integration of (9) yields (4). For the above current distribution in the domain Ω the Biot–Savart law is interpreted as the part of the magnetic field contributed by the currents in Ω after a subtraction of the magnetic field which arises from the currents outside of Ω.

2.2. An elliptic anisotropic boundary value problem

We now transform the above equations into an elliptic boundary value problem and solve it using the Lax–Milgram theorem. We will assume that Ω is simply connected, later we will also work with the assumption of convexity. Because of $\nabla \times E = 0$ there is an *electric potential* φ_E such that $E = \nabla \varphi_E$, i.e. for the current density j we have the equation

$$j = \sigma \nabla \varphi_E \tag{10}$$

We use the identity $\nabla\cdot\nabla\times A=0$, which is valid for an arbitrary sufficiently smooth vectorfield A, to derive from (2) the equation

$$\nabla\cdot j=\nabla\cdot\nabla\times H=0 \tag{11}$$

Now, using (10) we obtain the basic equation

$$\nabla\cdot\sigma\nabla\varphi_E=0 \quad \text{in } \Omega \tag{12}$$

for the electric potential φ_E. We will use the weak form

$$\int_\Omega \nabla\psi\cdot\sigma\nabla\varphi_E\,\mathrm{d}y=\int_{\partial\Omega}\psi\;\; v\cdot\sigma\nabla\varphi_E\,\mathrm{d}s \tag{13}$$

of this equation, which can be obtained for any function $\psi\in H^1(\Omega)$ in the Sobolev space $H^1(\Omega)$ (see Reference [12]) by integrating (12) and performing a partial integration. Here, the function $v\cdot\sigma\nabla\varphi_E$ is an element of the Sobolev space $H^{-1/2}(\partial\Omega)$, the right-hand side of (13) is understood in the sense of the dual space scalar product between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ and as in Section 2.1 we assume that the boundary is piecewise of class C^2 with well-behaved edges and corners such that the Gauss integral theorem can be applied. Given the normal component

$$v\cdot j=v\cdot\sigma\nabla\varphi_E=\tilde{g} \tag{14}$$

of j on the boundary $\partial\Omega$ of the domain Ω we obtain a Neumann problem with Equation (12) in Ω for the electric potential φ_E. We use the condition

$$\int_\Omega \varphi_E\,\mathrm{d}y=0 \tag{15}$$

to guarantee unique solvability of this problem.

Theorem 1
We assume that the conductivity tensor σ is coercive in Ω, i.e. there is a constant $c>0$ such that

$$\operatorname{Re} a\cdot\overline{\sigma a}\geqslant c|a|^2, \quad a\in\mathbb{R}^3 \tag{16}$$

Then, the boundary value problem given by (12)–(15) has a unique weak solution $\varphi_E\in H^1(\Omega)$ and this solution depends continuously on the boundary values $g\in H^{-1/2}(\partial\Omega)$.

Remark
Later we will use a diagonal tensor

$$\sigma=\begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{pmatrix} \tag{17}$$

for which coercivity is satisfied because of $\sigma_k>0$ in Ω.

Proof
The proof is well-known using either an equivalent minimization problem (see Reference [14]) or the Lax–Milgram theorem. Since we will later need parts of the proof to investigate a discrete model, we will briefly present the main steps. First, we show the uniqueness of the problem. Let φ be a solution of the homogeneous problem, i.e. a solution of (12), (14) with $\tilde{g}=0$. We have

$$\begin{aligned} 0 &= \int_\Omega \varphi \, \overline{\nabla \cdot \sigma \nabla \varphi} \, \mathrm{d}y \\ &= -\int_\Omega \nabla \varphi \cdot \overline{\sigma \nabla \varphi} \, \mathrm{d}y + \int_{\partial\Omega} \varphi \, v \cdot \overline{\sigma \nabla \varphi} \, \mathrm{d}s \end{aligned} \tag{18}$$

Because of the homogeneous boundary condition the second integral of (18) vanishes and we obtain

$$\int_\Omega \nabla \varphi \cdot \overline{\sigma \nabla \varphi} \, \mathrm{d}y = 0 \tag{19}$$

Since σ is coercive, this yields $\nabla\varphi = 0$ in Ω, i.e. $\varphi = c$ with some constant c. Finally from (15) we obtain $c=0$ and $\varphi \equiv 0$ in the domain Ω. This concludes the uniqueness proof.

We now use the theorem of Lax–Milgram to derive the existence of a solution to the boundary value problem. To this end we define the sesquilinear form

$$S(\psi, \varphi) := \int_\Omega \nabla \psi \cdot \overline{\sigma \nabla \varphi} \, \mathrm{d}y \tag{20}$$

and the bounded linear functional

$$F_g \psi := \int_{\partial\Omega} \psi \bar{g} \, \mathrm{d}s \tag{21}$$

on the space

$$X := H^1(\Omega) \cap \left\{ v : \int_\Omega v \, \mathrm{d}y = 0 \right\} \tag{22}$$

From (18) for the boundary value problem (12)–(14) and (15) we obtain the representation

$$S(\psi, \varphi_E) = F_g(\psi), \quad \psi \in X \tag{23}$$

According to the Riesz representation theorem there is a bounded linear operator $A: X \to X$ and a function $f \in X$ such that the above equation (23) can be written in the form

$$(\psi, A\varphi_E)_{H^1(\Omega)} = (\psi, f)_{H^1(\Omega)}, \quad \psi \in X \tag{24}$$

For the final step we need the coercivity of the sesquilinear form $S(\cdot,\cdot)$ in the norm of X, i.e. we need to show

$$\operatorname{Re} S(\varphi, \varphi) \geqslant c \|\varphi\|_X^2, \quad \varphi \in X \tag{25}$$

Coercivity (25) is not directly available, but it is possible to use the Poincaré inequalities of Theorem 2 (below) to derive from Equations (16) and (15) the estimate

$$\begin{aligned}
\operatorname{Re} S(\varphi,\varphi) &\geqslant c\,||\nabla\varphi||^2_{L^2(\Omega)} \\
&= \frac{c}{1+C}\,(||\nabla\varphi||^2_{L^2(\Omega)} + C||\nabla\varphi||^2_{L^2(\Omega)}) \\
&\geqslant \frac{c}{1+C}\,(||\nabla\varphi||^2_{L^2(\Omega)} + ||\varphi||^2_{L^2(\Omega)}) \\
&\geqslant \frac{c}{1+C}\,||\varphi||_{H^1(\Omega)}, \quad \varphi\in X
\end{aligned} \tag{26}$$

and thus coercivity (25). Now, the bounded invertibility of the operator A, the existence of a solution to Equation (23) and the continuous dependence of a solution on the right-hand side of the equation is a consequence of the theorem of Lax–Milgram, see Theorem 13.23 in Reference [15]. This shows the existence and continuity of the weak solution.

Finally, we remark that for proving the existence of a weak solution we did not use the condition

$$\int_{\partial\Omega} g\,\mathrm{d}s = 0 \tag{27}$$

on the boundary values g. But to obtain a strong solution from the weak equations this condition is needed for showing that $\nabla\cdot\sigma\nabla\varphi = 0$ from equation (28). □

For completeness and later use here we explicitly state the continuous version of the Poincaré inequality.

Theorem 2
Let Ω be a convex bounded domain in $\mathbb{R}^3$ and $u\in H^1(\Omega)$ with

$$\int_{\Omega} u\,\mathrm{d}s = 0 \tag{28}$$

Then there is a constant C such that

$$||u||_{L^2(\Omega)} \leqslant C||\nabla u||_{L^2(\Omega)} \tag{29}$$

Proof
See Chapter VII, 6 and 8 of Courant and Hilbert [16]. □

3. GRID MODEL AND ITS IMPLEMENTATION

We now describe the *grid model* due to the finite integration technique for the numerical solution of the continuous problem. This model can be realized physically, thus this proceeding opens up the opportunity to test the forward problem and inverse algorithms on real data using a physical realization of the grid model. Alternatively, finite element, finite volume or integral equation methods could be used for the numerical solution of the continuous forward problem—but these methods in general do not have a physical realization.

We will restrict our attention to the simplest possible case, i.e. we consider a rectangular domain $\Omega_\square$ given by

$$\Omega_\square = \left\{ y \in \mathbb{R}^3, \frac{-a_1}{2} < y_1 < \frac{a_1}{2}, \frac{-a_2}{2} < y_2 < \frac{a_2}{2}, \frac{-a_3}{2} < y_3 < \frac{a_3}{2} \right\} \tag{30}$$

with parameters $a_j > 0$, $j = 1, \ldots, 3$. We would like to remark that we were led to this simple geometry by some industrial application [5]. We denote the different parts of the surface $\partial\Omega_\square$ as follows:

$$\Gamma_1 := \left\{ y = \left(\frac{-a_1}{2}, y_2, y_3 \right) \in \partial\Omega_\square \right\}$$

$$\Gamma_2 := \left\{ y = \left(\frac{a_1}{2}, y_2, y_3 \right) \in \partial\Omega_\square \right\}$$

$$\Gamma_3 := \left\{ y = \left(y_1, \frac{-a_2}{2}, y_3 \right) \in \partial\Omega_\square \right\}$$

$$\Gamma_4 := \left\{ y = \left(y_1, \frac{a_2}{2}, y_3 \right) \in \partial\Omega_\square \right\}$$

$$\Gamma_5 := \left\{ y = \left(y_1, y_2, \frac{-a_3}{2} \right) \in \partial\Omega_\square \right\}$$

$$\Gamma_6 := \left\{ y = \left(y_1, y_2, \frac{a_3}{2} \right) \in \partial\Omega_\square \right\}$$

For the first four sides we use the abbreviation $\tilde{\Gamma} := \Gamma_1 \cup \cdots \cup \Gamma_4$.

3.1. Equations of the grid model

We consider the cuboid defined in (30) with the side surfaces $\tilde{\Gamma}$, the base surface Γ_5 and the top surface Γ_6. Currents are fed in at the base surface and are taken from the top surface, i.e. we have

$$v(x) \cdot j(x) = 0, \quad x \in \tilde{\Gamma} \tag{31}$$

and

$$e_3 \cdot j(x) \geqslant 0, \quad x \in \Gamma_5 \cup \Gamma_6 \tag{32}$$

For the grid model we use a regular grid $\mathscr{G}$ with n_1 points in the direction of the x_1-axis, n_2 points in the x_2-direction and n_3 points along the x_3-axis. We call the knot points

$$p_{klm} := (x_{1,k}, x_{2,l}, x_{3,m}) \tag{33}$$

for

$$k = 0, \ldots, n_1 - 1, \; l = 0, \ldots, n_2 - 1 \text{ and } m = 0, \ldots, n_3 - 1 \tag{34}$$

with

$$x_{s,\xi} := \frac{-a_s}{2} + \frac{\xi}{n_s - 1} a_s, \quad s = 1, 2, 3, \ \xi = 0, \ldots, n_s - 1 \tag{35}$$

In the discrete model currents may flow on the regular grid between the knot points. We denote the current flowing from a point p_{klm} to the following point:

$$p_{(k+1)lm}, \ p_{k(l+1)m} \text{ and } \ p_{kl(m+1)}$$

parallel to the x_1-, x_2- or x_3-axis by I_{klmx}, I_{klmy} or I_{klmz}, respectively. The virtual or real wire between the points is called s_{klmx}, s_{klmy} or s_{klmz}. For the index of the currents which are fed in the base surface we use the index $k = -1$. Thus the currents

$$I_{kl(-1)z} = I_{kl}^{\text{in}}, \ I_{kl(n_3-1)z} = I_{kl}^{\text{out}} \tag{36}$$

for $k = 0, \ldots, n_1 - 1$, $l = 0, \ldots, n_2 - 1$ are input parameters for the problem, where we have

$$I_{(n_1-1)lmx} = 0, \quad I_{k(n_2-1)my} = 0 \tag{37}$$

for k, l, m as in (34) because of the boundary condition (31). Analogously, currents with indices $k = -1$ and $l = -1$ are set equal to zero to model the boundary condition on the side surfaces.

Definition 3
The boundary currents used in (36) are called admissible, if the *conservation law*

$$\sum_{kl} I_{kl}^{\text{in}} = \sum_{kl} I_{kl}^{\text{out}} \tag{38}$$

is satisfied.

To each grid point we attach the cell $\mathscr{C}_{klm}$ defined by its corner points

$$\{ p_{(k+\delta_1)(l+\delta_2)(m+\delta_3)}, \delta_1, \delta_2, \delta_3 \in \{0, 1\} \} \tag{39}$$

For the discrete model we assume that the resistance of the wire $s_{klm\xi}$ for $\xi \in \{x, y, z\}$ is given by a positive real number $R_{klm\xi}$. The voltage between the knots of the grid is denoted by $U_{klm\xi}$. The current $I_{klm\xi}$, the resistance $R_{klm\xi}$ and the voltage $U_{klm\xi}$ satisfy *Ohm's law*

$$U_{klm\xi} = I_{klm\xi} \cdot R_{klm\xi} \tag{40}$$

(see for example Reference [17]). We have the classical conservation equation

$$I_{(k-1)lmx} + I_{k(l-1)my} + I_{kl(m-1)z} = I_{klmx} + I_{klmy} + I_{klmz} \tag{41}$$

for k, l, m sd in (34), i.e. the sum of incoming and outgoing currents is zero. The mesh theorem states that the sum of the voltages over each closed path is zero. A complete set of *mesh equations* is given by the elementary meshes of $\mathscr{G}$, defined by adjacent points, for example

$$p_{klm}, p_{(k+1)lm}, p_{(k+1)(l+1)m} \text{ and } \ p_{k(l+1)m} \tag{42}$$

The mesh equation for this example is

$$U_{klmx} + U_{(k+1)lmy} - U_{k(l+1)mx} - U_{klmy} = 0 \tag{43}$$

A complete and linearly independent set of mesh equations is given by the following set of equations. We have

$$U_{klmy} + U_{k(l+1)mz} - U_{kl(m+1)y} - U_{klmz} = 0 \tag{44}$$

for

$$k = 0, \dots, n_1 - 1,\ l = 0, \dots, n_2 - 2 \text{ and } m = 0, \dots, n_3 - 2 \tag{45}$$

$$U_{klmx} + U_{(k+1)lmz} - U_{kl(m+1)x} - U_{klmz} = 0 \tag{46}$$

for

$$k = 0, \dots, n_1 - 2,\ l = 0, \dots, n_2 - 1 \text{ and } m = 0, \dots, n_3 - 2 \tag{47}$$

and at the top surface the equations

$$U_{klmx} + U_{(k+1)lmy} - U_{k(l+1)mx} - U_{klmy} = 0 \tag{48}$$

for $k = 0, \dots, n_1 - 2$ and $l = 0, \dots, n_2 - 2$ and $m = n_3 - 1$. Thus, the full discrete grid model is given by Equations (36), (37), (41) and (44)–(48). These are

$$\begin{aligned} &\underbrace{n_1 n_2}_{(36)} + \underbrace{n_2 n_3 + n_1 n_3}_{(37)} + \underbrace{n_1 n_2 n_3}_{(41)} \\ &\quad + \underbrace{n_1(n_2 - 1)(n_3 - 1)}_{(44)} + \underbrace{(n_1 - 1) n_2 (n_3 - 1)}_{(46)} + \underbrace{(n_1 - 1)(n_2 - 1)}_{(48)} \\ &\quad = 3 n_1 n_2 n_3 + 1 \end{aligned} \tag{49}$$

equations, i.e. we have $3n_1n_2n_3 + 1$ equations for the $3n_1n_2n_3$ currents $I_{klm\xi}$. There is one redundant equation due to the admissibility condition for the prescribed currents. We can drop one of the knot equations and choose the last knot equation in (37) with the indices $k = n_1,\ l = n_2,\ m = n_3$ to set up the full equation system. For admissible boundary currents this last equation is a consequence of the other equations and the admissibility condition, which we will use in the uniqueness proof.

3.2. The grid model and the continuous model

First we interpret the grid model as a discretized version of the continuous model. To this end we successively derive Equations (36), (37), (41) and (44)–(48).

Equation (36) for the currents at the base and top surfaces is obtained directly as a discrete version of the boundary conditions for incoming and outgoing currents. The same is true for the other parts of the boundary conditions given by Equation (37).

The knot theorem (41) is the discrete version of (11) in its integral form

$$\int_{\mathcal{O}} v \cdot j \,\mathrm{d}s = 0 \tag{50}$$

for any closed surface $\mathcal{O} \subset \Omega$.

The mesh equations (44)–(48) are the discrete version of the equation

$$\nabla \times \nabla \varphi = 0$$

or its integral form

$$\int_{\mathscr{L}} \nabla \varphi \cdot \mathrm{d}\mathbf{l} = 0 \tag{51}$$

for any closed path $\mathscr{L}$ in Ω, where we write $\varphi = \varphi_E$.

In our case, the grid model has a practical physical realization and is of interest in itself. But when we use the grid model as an approximation for the continuous case we need to investigate the convergence of the solution of the discrete model towards the solution of the continuous model.

First, we will study an extended version of Equations (36), (37), (41) and (44)–(48), where we add a perturbation δ_ξ numbered successively by $\xi = 0, \ldots, n_1 n_2 n_3 - 1$, to the right-hand side of each of the equations. This will not change the non-singular matrix under consideration and the unique solvability of the system remains valid. When we write the original system as a linear equation for the current vector I in the form

$$AI = b \tag{52}$$

with the matrix A and the right-hand side b consisting out of I^{in}, I^{out} and zeros, then the modified system has the form

$$AI = b + \delta \tag{53}$$

The vector δ consists of two parts, the first of which defines a function on the boundary of the domain and corresponds to the boundary conditions (36), (37). The second part of δ can be interpreted as a function in Ω and corresponds to the knot theorem (41) and mesh Equations (44)–(48).

Given a current distribution $j \in C^2(\Omega) \cap C^1(\partial\Omega)$ let J_{n_1,n_2,n_3} be the vector which is obtained as the restriction of j to the nodes of the grid $\mathscr{G}$. We define the standard step function j_{n_1,n_2,n_3} on Ω by extending these values from each point p_{klm} as constants into the adjacent cell $\mathscr{C}_{klm}$. For a step function j_{n_1,n_2,n_3} or a vector I, respectively, we use the L^2-norm

$$\|I\|_{\mathscr{G}} := \left\{ \frac{a_1 a_2 a_3}{n_1 n_2 n_3} \sum_{klm} \left(R_{1,(k-1)lm} I^2_{1,(k-1)lm} + R_{2,k(l-1)m} I^2_{2,k(l-1)m} + R_{3,kl(m-1)} I^2_{3,kl(m-1)} \right) \right\}^{1/2} \tag{54}$$

On the boundary we use the norm

$$||a||_{\partial\mathcal{G}} := \left(\frac{a_2 a_3}{n_2 n_3} \sum_{p_{klm} \in \Gamma_1 \cup \Gamma_2} |a_{klm}|^2 + \frac{a_1 a_3}{n_1 n_3} \sum_{p_{klm} \in \Gamma_3 \cup \Gamma_4} |a_{klm}|^2 + \frac{a_1 a_2}{n_1 n_2} \sum_{p_{klm} \in \Gamma_5 \cup \Gamma_6} |a_{klm}|^2 \right)^{1/2} \quad (55)$$

For a function which is defined on parts of the boundary we extend the function by zero and use (55).

3.3. The discrete vector calculus

We will now introduce the notation of *discrete vector analysis* on a regular grid $\mathcal{G}$ given by (33) in a rectangular domain (30) and establish a discrete version of the Gauss integral theorem which is used to prove the solvability of the grid model and the convergence of the solution of this model towards the solution of the continuous model. A discrete differentiation operator in one dimension is defined by

$$(\partial\varphi)_l := \frac{1}{h}(\varphi_l - \varphi_{l-1}) \quad (56)$$

where h is the distance between two successive points of the regular grid. The discrete *gradient* in three dimensions is defined analogously by

$$(\nabla\varphi)_{klm} := \begin{pmatrix} \frac{n_1}{a_1}(\varphi_{klm} - \varphi_{(k-1)lm}) \\ \frac{n_2}{a_2}(\varphi_{klm} - \varphi_{k(l-1)m}) \\ \frac{n_3}{a_3}(\varphi_{klm} - \varphi_{kl(m-1)}) \end{pmatrix} \quad (57)$$

for k, l, m as in (34). Here, φ is a vector defined on the grid $\mathcal{G}$. The discrete *divergence* is defined for vector fields A, which are defined on the grid $\mathcal{G}$. We use

$$(\nabla \cdot A)_{klm} := \frac{n_1}{a_1}(A_{1,klm} - A_{1,(k-1)lm}) + \frac{n_2}{a_2}(A_{2,klm} - A_{2,k(l-1)m}) + \frac{n_3}{a_3}(A_{3,klm} - A_{3,kl(m-1)}) \quad (58)$$

for k, l, m as in (34). Now, the *Gauss integral theorem* obtains the special form

$$\frac{a_1 a_2 a_3}{n_1 n_2 n_3} \sum_{k=1}^{n_1-1} \sum_{l=1}^{n_2-1} \sum_{m=1}^{n_3-1} (\nabla \cdot A)_{klm} = \frac{a_2 a_3}{n_2 n_3} \sum_{l=1}^{n_2-1} \sum_{m=1}^{n_3-1} (A_{1,(n_1-1)lm} - A_{1,0lm})$$
$$+ \frac{a_1 a_3}{n_1 n_3} \sum_{k=1}^{n_2-1} \sum_{m=1}^{n_3-1} (A_{2,k(n_2-1)m} - A_{2,k0m})$$
$$+ \frac{a_1 a_2}{n_1 n_2} \sum_{k=1}^{n_1-1} \sum_{l=1}^{n_2-1} (A_{3,kl(n_3-1)} - A_{3,kl0}) \quad (59)$$

A proof of the discrete Gauss integral theorem is obtained by a simple reordering of the terms in the finite sum on the left-hand side of (59). As a discrete version of the *chain rule* we use the equation

$$(\partial(\varphi\psi))_l = \frac{1}{h}(\varphi_l \psi_l - \varphi_{l-1}\psi_{l-1})$$
$$= \varphi_l(\partial\psi)_l + (\partial\varphi)_l \psi_{l-1}$$

In particular, we obtain

$$(\nabla \cdot (\varphi A))_{klm} = \varphi_{klm}(\nabla \cdot A)_{klm} + (\partial_1\varphi)_{klm} A_{(k-1)lm}$$
$$+ (\partial_2\varphi)_{klm} A_{k(l-1)m} + (\partial_3\varphi)_{klm} A_{kl(m-1)} \quad (60)$$

We need to state a discrete version of the Poincaré inequality Theorem 2 with the norms used in Section 3.2.

Theorem 4
Let u be a function defined on the grid $\mathscr{G}$ with

$$\sum_{p_{klm}} u_{klm} = 0 \quad (61)$$

Then we have

$$\|u\|_{\mathscr{G}} \leqslant c' \|\nabla u\|_{\mathscr{G}} \quad (62)$$

with the norms defined in (54) and (55) and a constant c' not depending on the grid parameters n_1, n_2 and n_3.

Proof
A proof can be performed analogously to the proof of (1) in Reference [4, p. 488]. □

3.4. Solvability and convergence

We are now prepared to the prove uniqueness and subsequently the convergence properties of the scheme.

Theorem 5 (*Unique solvability of the finite system*)
The system of equations defined by (36), (37), (40), (41) and (44)–(48) is uniquely solvable for each admissible set (36) of incoming and outgoing currents.

Proof
We need to show that the quadratic matrix arising from the equations is non-singular. To this end it is sufficient to show its injectivity. We will now develop a proof using a discrete form of the uniqueness part of Theorem 1. First, we note that with the discrete vector analysis notation we have

$$(\nabla \cdot I)_{klm} = 0 \tag{63}$$

for k, l, m as in (34). This is just another formulation of the knot theorem (41). Because of the mesh equations there is a potential φ on the grid $\mathscr{G}$ such that

$$(\nabla\varphi)_{klm} = \begin{pmatrix} U_{1,(k-1)lm} \\ U_{2,k(l-1)m} \\ U_{3,kl(m-1)} \end{pmatrix} \tag{64}$$

We now use the discrete form of the Gauss theorem (59) applied to the vector $\nabla \cdot (\varphi I)$ with the potential φ and the current vector I on the grid $\mathscr{G}$ to obtain

$$\begin{aligned} 0 &= \frac{a_1 a_2 a_3}{n_1 n_2 n_3} \sum_{klm} \varphi_{klm}\, (\nabla \cdot I)_{klm} \\ &= -\frac{a_1 a_2 a_3}{n_1 n_2 n_3} \sum_{klm} ((\partial_1\varphi)_{klm} I_{1,(k-1)lm} + (\partial_2\varphi)_{klm} I_{2,k(l-1)m} \\ &\quad + (\partial_3\varphi)_{klm} I_{3,kl(m-1)}) \\ &\quad + \frac{a_2 a_3}{n_2 n_3} \sum_{l=1}^{n_2-1} \sum_{m=1}^{n_3-1} (\varphi_{(n_1-1)lm} I_{1,(n_1-1)lm} - \varphi_{0lm} I_{1,0lm}) \\ &\quad + \frac{a_1 a_3}{n_1 n_3} \sum_{k=1}^{n_2-1} \sum_{m=1}^{n_3-1} (\varphi_{k(n_2-1)m} I_{2,k(n_2-1)m} - \varphi_{k0m} I_{2,k0m}) \\ &\quad + \frac{a_1 a_2}{n_1 n_2} \sum_{k=1}^{n_1-1} \sum_{l=1}^{n_2-1} (\varphi_{kl(n_3-1)} I_{3,kl(n_3-1)} - \varphi_{kl0} I_{3,kl0}) \end{aligned} \tag{65}$$

If I satisfies homogeneous boundary conditions on $\partial\Omega$, then the last three terms of (65) vanish. Because of

$$(\partial_1\varphi)_{klm} = U_{1,(k-1)lm} = R_{1,(k-1)lm} I_{1,(k-1)lm} \tag{66}$$

and analogous equations for the second and third components we obtain

$$0 = \frac{a_1a_2a_3}{n_1n_2n_3}\sum_{klm}(R_{1,(k-1)lm}I^2_{1,(k-1)lm} + R_{2,k(l-1)m}I^2_{2,k(l-1)m} + R_{3,kl(m-1)}I^2_{3,kl(m-1)}) \tag{67}$$

Since all resistance components are positive, we obtain $I \equiv 0$ on $\mathscr{G}$ and therefore the injectivity and regularity of the matrix arising from Equations (36), (37), (40), (41) and (44)–(48). □

Theorem 6 (*Stability*)
For a solution of Equation (53) we have

$$||I||_{\mathscr{G}} \leqslant C(||I^{\text{in}}||_{\partial\mathscr{G}} + ||I^{\text{out}}||_{\partial\mathscr{G}} + ||\delta||_{\mathscr{G}}) \tag{68}$$

with a constant C not depending on n_1, n_2 and n_3.

Proof
Let I be a solution of (53). Then we define U by (40) and the potential φ by (66) with the additional condition

$$\sum_{p_{klm}} \varphi_{klm} = 0 \tag{69}$$

For the following estimates we first remark that because of $R_{klm} \geqslant \eta > 0$ with some constant η we have

$$\left(\sum_{\xi} |\delta_\xi|^2\right)^{1/2} \leqslant \frac{1}{\eta}\,||\delta||_{\mathscr{G}} \tag{70}$$

Second, for points p_{klm} in the interior of Ω we have $(\nabla\cdot I)_{klm} = \delta_\xi$ for some ξ where to each interior node of $\mathscr{G}$ there is exactly one δ_ξ as organized by system (53). On the boundary we have equations of the form $I_{kl(-1)z} = I^{\text{in}}_{kl} + \delta_\xi$ and $I_{(n_1-1)lmx} = \delta_\xi$ for some ξ arising from (36) and (37), where now to each point there is exactly one δ_ξ not used for the interior points. We now proceed as in (65)–(67) to derive

$$\begin{aligned} ||I||^2_{\mathscr{G}} &= \frac{a_1a_2a_3}{n_1n_2n_3}\sum_{klm}(R_{(k-1)lm}I^2_{(k-1)lm} \\ &\quad + R_{k(l-1)m}I^2_{k(l-1)m} + R_{kl(m-1)}I^2_{kl(m-1)}) \\ &= -\frac{a_1a_2a_3}{n_1n_2n_3}\sum_{klm}\varphi_{klm}\,(\nabla\cdot I)_{klm} \end{aligned}$$

$$
\begin{aligned}
&+\frac{a_2a_3}{n_2n_3}\sum_{l=1}^{n_2-1}\sum_{m=1}^{n_3-1}(\varphi_{(n_1-1)lm}I_{1,(n_1-1)lm}-\varphi_{0lm}I_{1,0lm})\\
&+\frac{a_1a_3}{n_1n_3}\sum_{k=1}^{n_1-1}\sum_{m=1}^{n_3-1}(\varphi_{k(n_2-1)m}I_{2,k(n_2-1)m}-\varphi_{k0m}I_{2,k0m})\\
&+\frac{a_1a_2}{n_1n_2}\sum_{k=1}^{n_1-1}\sum_{l=1}^{n_2-1}(\varphi_{kl(n_3-1)}I_{3,kl(n_3-1)}-\varphi_{kl0}I_{3,kl0})\\
\leqslant\ & c||\varphi||_{\mathscr{G}}(||\delta||_{\mathscr{G}}+||I^{\text{in}}||_{\partial\mathscr{G}}+||I^{\text{out}}||_{\partial\mathscr{G}}) \qquad (71)
\end{aligned}
$$

with some constant c not depending on n_1, n_2 and n_3. We use the discrete form Theorem 4 of the Poincaré estimate to derive

$$||\varphi||_{\mathscr{G}}\leqslant c'||I||_{\mathscr{G}} \qquad (72)$$

with some constant c' and finally obtain (68) with $C=cc'$. □

Theorem 7 (*Consistency and Convergence*)
Given some solution $j\in C^1(\Omega)\cap C(\bar{\Omega})$ of the continuous boundary value problem (12), (14) and (15) and a grid $\mathscr{G}$ let I_{true} be the discretized version of j and I be the solution of the grid model with boundary values given by $v\cdot j$, i.e. $I_{\text{true}}:=j(p_{klm})$ and $AI=b$. Then with

$$h:=\max\left\{\frac{a_1}{n_1},\frac{a_2}{n_2},\frac{a_3}{n_3}\right\} \qquad (73)$$

we have

$$||I_{\text{true}}-I||_{\mathscr{G}}=O(h) \qquad (74)$$

for sufficiently small h. This proves (linear) convergence of the solution of the grid model to the solution of the continuous model.

Proof
On the grid $\mathscr{G}$ we approximate the integrals (50) and (51) for the continuous solution j by the rectangular rule and obtain the system of equations (53). Here, the components of δ are $O(h)$ due to the linear convergence of the rectangular rule for differentiable functions. This is the *consistency* of the discretization scheme and it is usually used as a key argument to derive the convergence of the finite integration technique. The solution I solves (53), and by subtracting the two Equations (52) and (53) we now obtain

$$A(I_{\text{true}}-I)=\delta \qquad (75)$$

and thus estimate (74). □

3.5. Implementation and examples

For the numerical solution of the direct problem according to the grid model a program package in MATLAB has been developed. Given a resistance function, parameters for the rectangular domain under consideration and the currents at the boundary of the domain, the currents in the domain and the magnetic field on some external cylinder is calculated. All functions have been combined into a function magneticfield.

```
>> [Bexakt,Jexakt]=magneticfield(a, n, b, N, m, Rf)
```

with

```
input parameters:

   a   length of the axis of the cuboid a=[a1,a2,a3]
   n   number of discretization points along the three axis
       n=[n1,n2,n3]
   b   radius and height b=[r,h] of measurement cylinder
   N   number of discretization points N=[nr,nz] for the
       measurement cylinder
   m   parameter for current boundary input
       1 : uniform input on base and uniform outflow on top
       2 : in and outflow of current centered
   Rf  Name (string) of a resistance function
       of the form Rf(x,y,z) (filename Rf.m)
       giving the resistance in the point x,y,z as vector
       [Rx,Ry,Rz].

output parameters:

   Jexakt   current in the cuboid, vector of size 3*n1*n2*n3
   Bexakt   magnetic field on the measurement cylinder
            of dimension 3*nr*nz
```

The program solves the linear system arising from Equations (36), (37), (41) and (44)–(48) and calculates the magnetic field B according to the Bio–Savart law (4) using a simple rectangular rule. A graphical representation of the calculated current density is obtained using the function

```
>> stromvektorplot(a, n, J)
```

with the

```
input parameters:

   a   length of the axis of the cuboid a=[a1,a2,a3]
   n   number of discretization points along the three axis
       n=[n1,n2,n3].
   J   current in the cuboid, vector of size 3*n1*n2*n3
```

The figure which is produced by stromvektorplot can be supplemented by a graph of the magnetic field in vectorial form. This is done by the function

```
>> BFeldvektorplot(b, N, B)
```


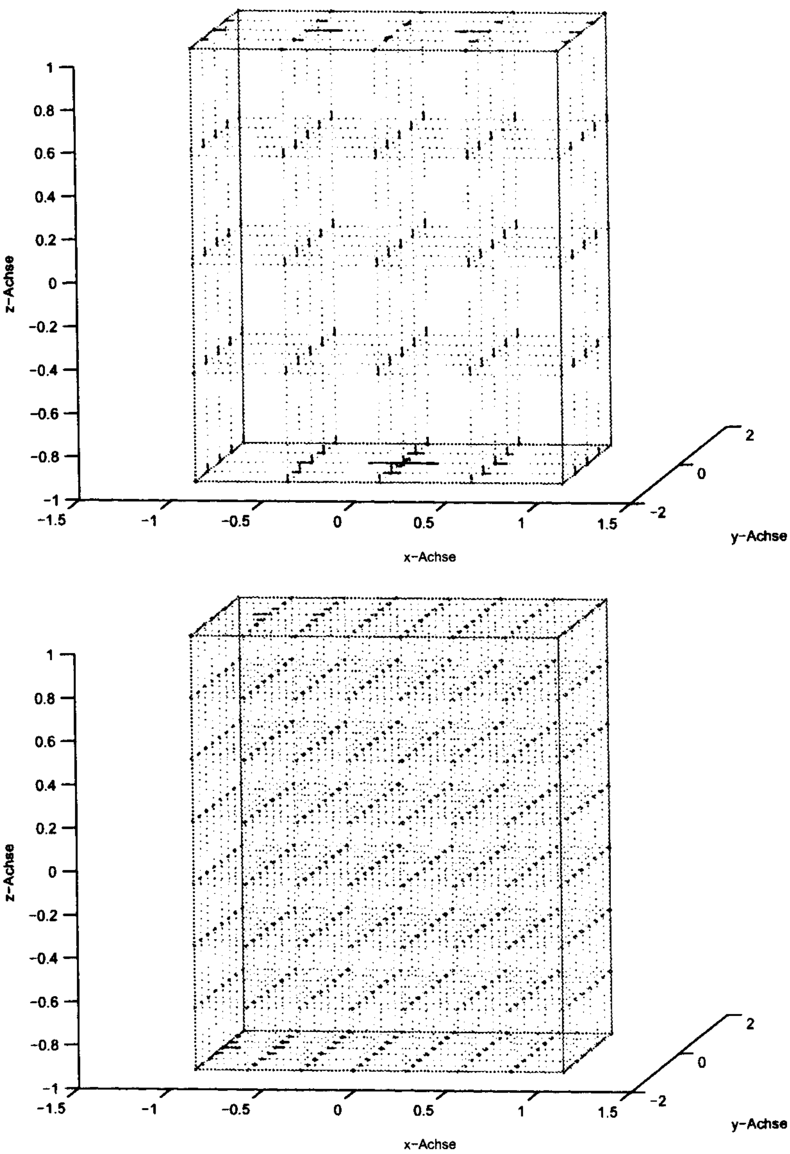

Figure 1. Grids with 5 and 8 grid points in each direction, i.e. $n=[5,5,5]$ or $n=[8,8,8]$, and currents (blue arrows) for a homogeneous resistance distribution and centred input/outflow.

where we have the

```
input parameters:

  b   radius and height b=[r,h] of measurement cylinder
  N   number of discretization points N=[nr,nz] for the measurement cylinder
  B   magnetic field on the measurement cylinder with dimension 3*nr*nz
```

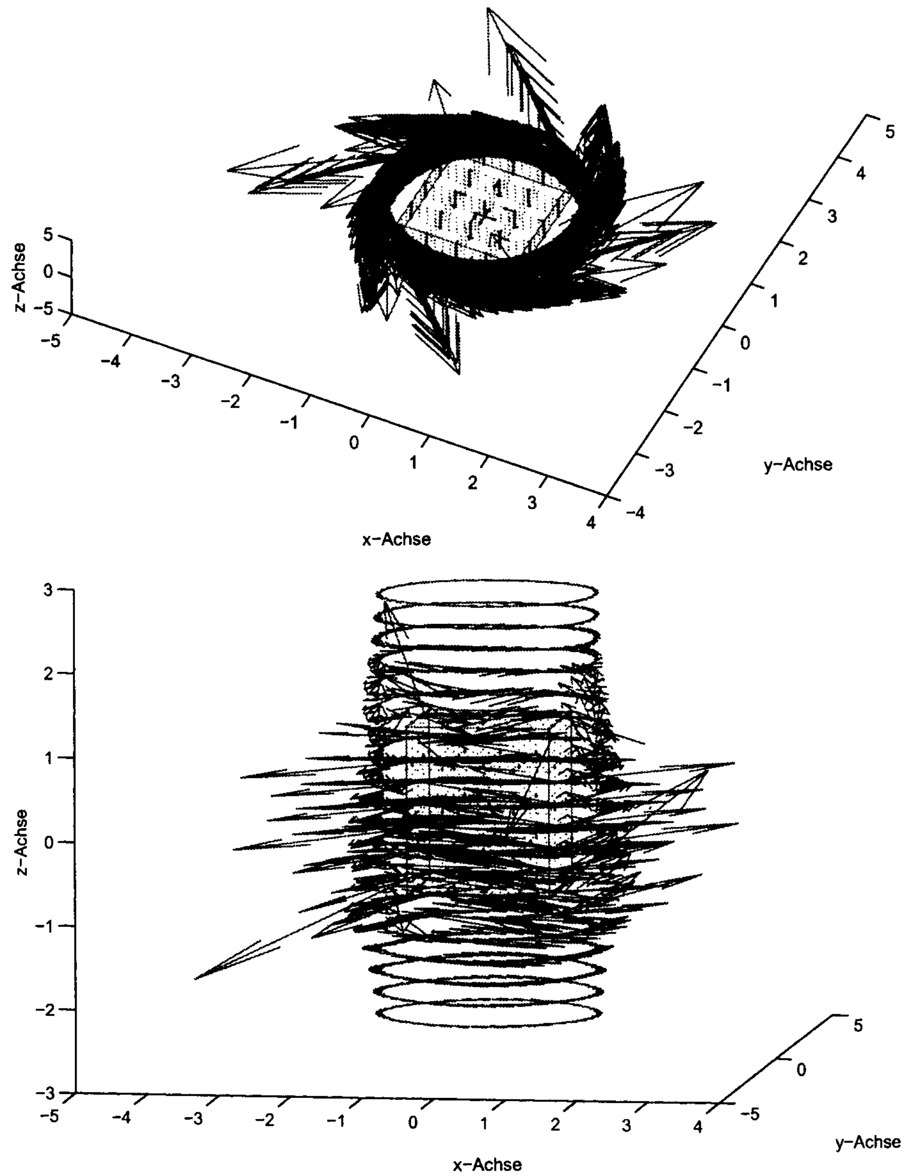

Figure 2. The grid with $n=[5,5,5]$ and the magnetic field on a measurement cylinder for some homogeneous current distribution with two different views performed by rotation using MATLAB.

Examples for the output are shown in Figures 1 and 2. We show current density distributions with 5 or 8 discretization points in each direction, respectively. The figures can be interactively turned using the MATLAB system and provide a visual check of the program for the direct problem as well as a tool for checking measurements when the inverse problem is investigated and applied to real data.

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