INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **21** (2005) 955–967 doi:10.1088/0266-5611/21/3/010

On uniqueness and non-uniqueness for current reconstruction from magnetic fields

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Received 2 December 2004, in final form 1 March 2005
Published 11 April 2005
Online at stacks.iop.org/IP/21/955

Abstract
The goal of this paper is to provide a basis for the analysis of the limits of the reconstructability of current densities from their magnetic fields as used for non-destructive testing and monitoring of fuel cells. For the reconstruction of a current density from its magnetic field, we study the properties of the Biot-Savart operator W. In particular, the nullspace $N(W)$ of the Biot–Savart operator and its orthogonal space $N(W)^{\perp}$ with respect to the L^2 scalar product are characterized. The characterization of these spaces is a basic step for the evaluation of the principal limits of magnetic tomography for fuel cells and for the development of efficient reconstruction algorithms. Further, practically realizable examples for elements in the nullspace $N(W)$ are provided. Finally, for a discrete wire network we show uniqueness for current reconstructions, i.e. the result $N(W) = \{0\}$.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Magnetic tomography is concerned with the reconstruction of currents or current densities from their magnetic fields. The magnetic field H of a current density distribution j in some domain B is described by the Biot–Savart law

$$H(x) = \operatorname{curl} \int_B \Phi(x, y) j(y) \, \mathrm{d}s(y), \qquad x \in \mathbb{R}^3 \tag{1.1}$$

with

$$\Phi(x, y) := \frac{1}{4\pi} \frac{1}{|x - y|}, \qquad x \neq y \in \mathbb{R}^3. \tag{1.2}$$

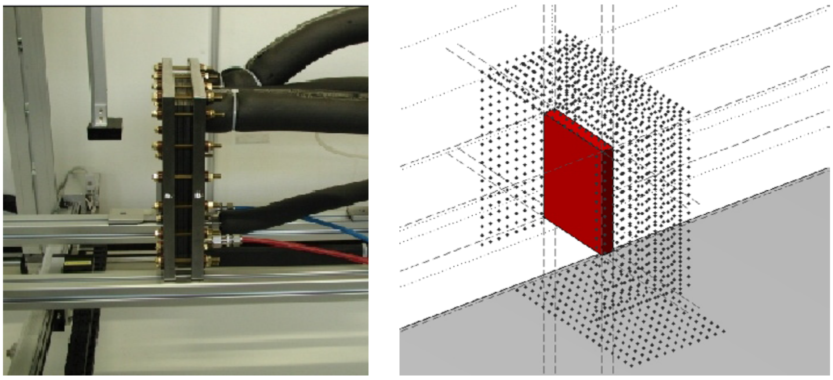

Figure 1. Fuel cell with magnetic measurement device (left) and measurement set-up for MagneTom measurement and reconstruction software (right). (Reproduced with permission of the Research Centre Jülich, Germany and the TomoScience GbR, Wolfsburg, Germany.)

With measurements H_{meas} of H on some outer surface $\Lambda \subset \mathbb{R}^3 \setminus \overline{B}$ the reconstruction of j needs to solve the integral equation

$$Wj = H_{\text{meas}} \quad \text{on } \Lambda, \tag{1.3}$$

with the Biot–Savart integral operator

$$(Wj)(x) = \operatorname{curl} \int_B \Phi(x, y) j(y) \, \mathrm{d}y, \qquad x \in \mathbb{R}^3. \tag{1.4}$$

Here, we focus on the *fuel cell* application, i.e. the reconstruction on current densities in a fuel cell from measurements of the magnetic field in the exterior of the cell. Usually fuel cells consist of rectangular areas of different layers with end plates of cuboid form. The outer measurement surface Λ can be chosen by the measurement device. It is usually well separated from the cell area B. Thus, we assume that the domain B where currents flow is a bounded domain with Lipschitz continuous boundary and that the surface Λ is sufficiently smooth. We will assume that the measurements of H on Λ or particular components of H on Λ uniquely determine H in the exterior of B.

The constructability of j from H_{meas} has already been studied by Kress, Kühn and Potthast in [4]. In this work the nullspace of the operator W has been shown to contain the set

$$M := \left\{ j = \triangle m \colon \, m \in \mathbf{C}_0^2(B) \right\}. \tag{1.5}$$

Further, it is well known that the classical Tikhonov regularization

$$j^{(\alpha)} := (\alpha I + W^* W)^{-1} W^* H_{\text{meas}} \tag{1.6}$$

for the approximate reconstruction of j from H_{meas} in the limit $\alpha \to 0$ for exact data reconstructs a projection $j_p = P(j)$ of the original current density j onto the space $N(W)^{\perp}$ perpendicular to the nullspace $N(W)$ of W. Without further *a priori* knowledge magnetic tomography can at most reconstruct the reconstruction j_p onto the perpendicular space $N(W)^{\perp}$. This establishes a basic limit to the error bounds for magnetic tomography. Kress *et al* showed by numerical experiments that the density j_p still contains important features of the original density j. Thus they proved that magnetic tomography is a reasonable approach and that even with the non-trivial nullspace $N(W)$ magnetic tomography is possible.

The central goal of this paper is to provide a complete characterization of the spaces $N(W)$ and $N(W)^{\perp}$ and to illustrate the situation with some generic examples. First, this characterization can be used as a basis for more efficient reconstruction schemes which take into account the special form of $N(W)^{\perp}$. Second, we can use the space $N(W)^{\perp}$ to give precise estimates about the best possible reconstruction of current densities and, thus, provide a method to both theoretically and numerically evaluate the limits of magnetic tomography. With knowledge of $N(W)^{\perp}$ we can calculate the orthogonal projector $P : L^2(B) \to N(W)^{\perp}$. Then, the minimal error for the reconstruction of j from $H = Wj$ is given by

$$E(j) = \|j - P(j)\|_{L^2(B)}. \tag{1.7}$$

The numerical evaluation of this estimate for fuel cell applications is an important problem for future work.

In section 2 we collect notation, basic definitions and properties of the Biot–Savart operator. In section 3 the nullspace $N(W)$ and its orthogonal complement $N(W)^{\perp}$ are studied and characterized. In section 4 we provide examples for nullspace and non-nullspace elements and also prove uniqueness for current reconstructions in the case of a discrete wire network.

2. Properties of the Biot–Savart operator

This section serves to collect basic definitions and properties of the Biot–Savart integral operator. By $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\pm 1}$ we denote the scalar products in L^2 and $H^{\pm 1}$ or its vectorial versions. For a field $u \in L^2(B)$ the weak divergence is defined as the element $\operatorname{div} u$ of $H^{-1}(B)$ with

$$\int_B \phi \operatorname{div} u \,\mathrm{d}x = \int_B u \cdot \operatorname{grad} \phi \,\mathrm{d}x, \qquad \forall\, \phi \in H_0^1(B). \tag{2.1}$$

We will work with the spaces

$$\begin{aligned} H_{\mathrm{div}}(B) &:= \{v \in L^2(B)\colon \operatorname{div} v \in L^2(B)\}, \\ H_{\mathrm{div}=0}(B) &:= \{v \in H_{\mathrm{div}}(B)\colon \operatorname{div} v = 0\}, \end{aligned} \tag{2.2}$$

where for the second space $H_{\mathrm{div}=0}(B)$ we employ the L^2 scalar product.

For boundary value problems with Lipschitz domains we need to define appropriate *trace operators* and collect their properties.

Definition 2.1. *For a domain B with Lipschitz continuous boundary and a sufficiently smooth vector field u the trace operators*

$$\gamma_0[B]\colon\ u \mapsto u_{|\partial B} \qquad \gamma_\nu[B]\colon\ u \mapsto \nu \cdot u_{|\partial B}$$

can be defined. The operator γ_ν is a bounded linear operator

$$\gamma_\nu[B]\colon\ H_{\mathrm{div}}(B) \to H^{-\frac{1}{2}}(\partial B). \tag{2.3}$$

The operator γ_0 can be defined as bounded linear operator

$$\gamma_0[B]\colon\ H^1(B) \to H^{\frac{1}{2}}(\partial B). \tag{2.4}$$

Proof. A proof can be found in chapter I, section 2, theorem 2.5 of [6] for (2.3) and chapter I, section 1, theorem 1.5 for (2.4). □

Next, we need some notations and operators from potential theory. First, consider the *single layer potential*

$$(S\phi)(x) := \int_{\partial B} \Phi(x, y)\phi(y)\,\mathrm{d}s(y), \qquad x \in \mathbb{R}^3. \tag{2.5}$$

We use the same letter if the scalar density ϕ is replaced by a vector-valued density v. For the proof of the following result we refer to [2].

Theorem 2.2. *The single-layer potential S and its vector-valued version are well-defined mappings*

$$S : H^{-\frac{1}{2}}(\partial B) \to H^1(\mathbb{R}^3). \tag{2.6}$$

Second, we consider the *volume potential*

$$(Vj)(x) := \int_B \Phi(x, y) j(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^3 \tag{2.7}$$

for a vector field j in B or for a scalar function. Its basic properties are summarized in the following theorem.

Theorem 2.3. *Let B, G be bounded domains in $\mathbb{R}^3$. The volume potential V maps $L^2(B)$ continuously into $H^2(G)$. The function Vj is analytic in B_e. Moreover,*

$$\triangle Vj = 0 \quad \text{in } B_e, \tag{2.8}$$

$$\triangle Vj = -j \quad \text{in } B, \tag{2.9}$$

where the second equation has to be understood in the L^2-sense.

Proof. The proof is carried out in [1], theorems 8.1 and 8.2. □

As further preparation step, we will now show that the divergence of the volume potential for a divergence free current density j in some domain B can be represented by a single-layer potential over the boundary ∂B, i.e. it does depend only on the boundary values $\nu \cdot j$ of the current density under consideration.

Lemma 2.4. *Let $j \in H_{\mathrm{div}=0}(B)$ for some Lipschitz continuous domain $B \subset \mathbb{R}^3$. Then we obtain*

$$\mathrm{div}(Vj) = -S(\nu \cdot j) \quad \text{in } \mathbb{R}^3, \tag{2.10}$$

where $\nu \cdot j$ is to be understood in the sense of $\gamma_\nu[B]j$.

Proof. For a vector field $j \in H_{\mathrm{div}=0}(B)$ we have

$$\mathrm{div}(\Phi j) = (\mathrm{grad}\, \Phi) \cdot j. \tag{2.11}$$

Now, using $\mathrm{grad}_x \Phi(x, y) = -\mathrm{grad}_y \Phi(x, y)$ and Gauss' integral theorem we calculate

$$\begin{aligned}
\mathrm{div}(Vj)(x) &= \int_B \mathrm{div}_x(\Phi(x, y) j(y))\, \mathrm{d}y = \int_B \mathrm{grad}_x \Phi(x, y) \cdot j(y)\, \mathrm{d}y \\
&= -\int_B \mathrm{grad}_y \Phi(x, y) \cdot j(y)\, \mathrm{d}y = -\int_B \mathrm{div}_y(\Phi(x, y) j(y))\, \mathrm{d}y \\
&= -\int_{\partial B} \Phi(x, y)(\nu \cdot j(y))\, \mathrm{d}y = -S(\nu \cdot j)(x), \qquad x \in \mathbb{R}^3
\end{aligned}$$

and the proof is complete. □

The operator W is the curl of the operator V

$$Wj = \mathrm{curl}(Vj). \tag{2.12}$$

This leads to the following result.

Corollary 2.5. *Let B, G be bounded domains in $\mathbb{R}^3$. The operator W defined by (1.4) maps $L^2(B)$ boundedly into $H^1(G)$.*

As shown in lemma 2.4, for currents in some bounded domain B it is important to take into account the normal components $\nu \cdot j$, i.e. the currents which flow into or out of the area under consideration. To this end we introduce

$$(S^{\nabla} j)(x) := \operatorname{grad} S(\nu \cdot j)(x) = \operatorname{grad} \int_{\partial B} \Phi(x, y)(\nu \cdot j)(y)\, \mathrm{d}s(y). \tag{2.13}$$

The following result is a further preparation to study the nullspace of magnetic fields of current distributions.

Lemma 2.6. *For the Biot–Savart operator W with density $j \in H_{\mathrm{div}=0}(B)$ we have*

$$\operatorname{div} Wj = 0, \qquad \operatorname{curl} Wj = j - S^{\nabla} j \qquad \text{in } B \tag{2.14}$$

$$\operatorname{div} Wj = 0, \qquad \operatorname{curl} Wj = -S^{\nabla} j \qquad \text{in } B_e. \tag{2.15}$$

Proof. The statements for the divergence $\operatorname{div} Wj$ are obtained for the curl field $Wj = \operatorname{curl} Vj$ by

$$\operatorname{div} \operatorname{curl} = 0. \tag{2.16}$$

The statements for $\operatorname{curl} Wj$ can be derived analogously to the proof of equation (2.14) from [4], lemma 8 as follows. We consider current densities $j \in H_{\mathrm{div}=0}(B)$. Using

$$\operatorname{curl} \operatorname{curl} = -\triangle + \operatorname{grad} \operatorname{div} \tag{2.17}$$

for the volume potential Vj and $\operatorname{div} Vj = -S(\nu \cdot j)$ from theorem 2.4 we obtain

$$\begin{aligned} \operatorname{curl} Wj &= \operatorname{curl} \operatorname{curl} Vj \\ &= -\triangle Vj + \operatorname{grad} \operatorname{div} Vj \\ &= -\triangle Vj - \operatorname{grad} S(\nu \cdot j) \end{aligned} \tag{2.18}$$

in $\mathbb{R}^3$. In the domain B we can use $\triangle Vj = -j$ to derive

$$\begin{aligned} \operatorname{curl} Wj &= j - \operatorname{grad} S(\nu \cdot j) \\ &= j - S^{\nabla} j. \end{aligned} \tag{2.19}$$

This yields the second statement of (2.14) for a density in $H_{\mathrm{div}=0}(B)$. Using the equation (2.18) in the domain B_e with the help of $\operatorname{div} Vj = 0$ we calculate

$$\begin{aligned} \operatorname{curl} Wj &= -\operatorname{grad} S(\nu \cdot j) \\ &= -S^{\nabla} j, \end{aligned} \tag{2.20}$$

for $j \in H_{\mathrm{div}=0}(B)$. □

Maxwell's equations demand the relation

$$\operatorname{curl} H = j \tag{2.21}$$

between the magnetic field H and the current j. Thus, the physical magnetic field for a current distribution j in B is given by $H = Wj$ if and only if $\nu \cdot j = 0$ on ∂B, i.e. if we have a closed system.

3. The nullspace and its orthogonal complement

As discussed in the introduction, a basic problem for reconstructing current distributions is the non-trivial nullspace of the mapping W. We define $N(W)$ to be the space of functions $j \in H_{\mathrm{div}=0}(B)$ such that $Wj = 0$ on Λ. By the assumptions on Λ this is equivalent to the equation $Wj = 0$ in B_e. In particular, by an application of the trace operator we obtain

$$(Wj)(y) = 0, \qquad y \in \partial B. \tag{3.1}$$

In order to characterize the nullspace of W we start to derive some properties of its elements.

For the following lemma please note that for non-vanishing currents in the fuel-cell application the normal component $\nu \cdot j$ of the current density j does not vanish on the boundary ∂B of the fuel cell. It is a consequence of this lemma that any such current through a fuel cell will generate some magnetic field outside. There might be different currents which have the same magnetic field, but in this case their normal components on the boundary will coincide.

Lemma 3.1. *Let $j_0 \in N(W)$, then we have $\nu \cdot j_0 = 0$ on ∂B. In particular, for $j_0 \in N(W)$ we have*

$$S^{\nabla} j_0 = 0 \quad \text{in } B_e, \qquad \operatorname{curl}(W j_0) = j_0 \quad \text{in } B. \tag{3.2}$$

Proof. Consider a function $j_0 \in N(W)$, i.e. we have $Wj_0 = 0$ in B_e and also $\operatorname{curl}(Wj_0) = 0$ in B_e. Using (2.15) we calculate

$$-\operatorname{grad} S(\nu \cdot j_0) = -S^{\nabla} j_0 = \operatorname{curl} W j_0 = 0 \quad \text{in } B_e.$$

Therefore $S(\nu \cdot j_0)$ must be constant in B_e and from the behaviour of the single-layer potential at infinity we conclude $S(\nu \cdot j_0) = 0$ in B_e. The single-layer potential is continuous on ∂B. Thus, $S(\nu \cdot j_0)$ is a harmonic function in B with homogeneous Dirichlet boundary condition and has to be zero. From the jump relation for the normal derivative of the single-layer potential (compare [2]) we get $\nu \cdot j_0 = 0$ on ∂B. Inserting this into (2.14) we get (3.2). This ends the proof. □

As a further preparation we prove the following result.

Lemma 3.2. *For every $j_0 \in N(W)$ the magnetic field H defined by $H := Wj_0$ satisfies $H \in H_0^1(B)$ with* $\operatorname{div} H = 0$ *and* $\operatorname{curl} H = j_0$ *in B.*

Proof. First, we note for Wj_0 the traces $\gamma_0[B]Wj_0$ and $\gamma_0[B_e]Wj_0$ are identical on ∂B. This can be obtained by the following argument. The function $w := Wj_0$ is in $H^1(G)$ for some domain G such that $\overline{B} \subset G$ with smooth boundary ∂G. Then, $w|_B$ is in $H^1(B)$ and $w|_{G\backslash\overline{B}}$ is in $H^1(G\backslash\overline{B})$. For functions $\varphi \in C^1(G)$ the traces $\gamma_0[B]\varphi$ and $\gamma_0[G\backslash\overline{B}]\varphi$ are identical on ∂B. Since the trace operators are continuous

$$\begin{aligned} &\gamma_0[B]\colon\ H^1(B) \to H^{1/2}(\partial B) \\ &\gamma_0[G\backslash\overline{B}]\colon\ H^1(G\backslash\overline{B}) \to H^{1/2}(\partial(G\backslash\overline{B})) \end{aligned} \tag{3.3}$$

their restriction to ∂B is identical on a dense set, this restriction also coincides on $H^1(G)$. Now, the assumption $Wj_0 = 0$ in B_e yields $Wj_0 \in H_0^1(B)$.

Further, by equation (2.14) we have div $H = 0$ in B and with the help of lemma 3.1 we calculate

$$\operatorname{curl} H = \operatorname{curl} W j_0 = j_0 - \nabla S(\underbrace{\nu \cdot j_0}_{=0}) = j_0, \tag{3.4}$$

which completes the proof. □

Our first main result now characterizes the nullspace of W. We use the notation

$$X := \left\{\operatorname{curl} v\colon\ v \in H_0^1(B), \operatorname{div} v = 0\right\}. \tag{3.5}$$

Theorem 3.3. *The nullspace of* $W : H_{\mathrm{div}=0}(B) \to L^2(\Lambda)$ *is given by* X*, i.e.*

$$N(W) = X. \tag{3.6}$$

Proof. The inclusion $N(W) \subset X$ is a consequence of lemma 3.2, since j_0 is the curl of its magnetic field H which does satisfy the boundary condition $\gamma_0[B]H = 0$ and is divergence free.

We need to verify $N(W) \supset X$, i.e. that for an element $j \in X$ we have $Wj = 0$ on Λ. Consider a vector field $v \in H_0^1(B)$ with div $v = 0$ in B. We will need the formula

$$\operatorname{curl}(\Phi(x, y)v(y)) = \nabla\Phi(x, y) \times v(y) + \Phi(x, y) \operatorname{curl} v(y), \tag{3.7}$$

which is true for both differentiations with respect to x and y. By Stokes theorem we obtain

$$\int_B \operatorname{curl}_y(\Phi(x, y)v(y))\,\mathrm{d}y = \int_{\partial B} \Phi(x, y) \underbrace{\nu \times v(y)}_{=0 \text{ on } \partial B} \mathrm{d}s(y) = 0. \tag{3.8}$$

Now we calculate

$$\begin{aligned} W(\operatorname{curl} v)(x) &= \operatorname{curl} \int_B \Phi(x, y) \operatorname{curl} v(y)\,\mathrm{d}y \\ &\overset{(3.7)}{=} \operatorname{curl} \int_B \operatorname{curl}_y(\Phi(x, y)v(y))\,\mathrm{d}y - \operatorname{curl} \int_B (\nabla_y\Phi(x, y)) \times v(y)\,\mathrm{d}y \end{aligned} \tag{3.9}$$

which by an application of (3.8) and $\nabla_x\Phi(x, y) = -\nabla_y\Phi(x, y)$ can be transformed into

$$\begin{aligned} W(\operatorname{curl} v)(x) &= \operatorname{curl} \int_B (\nabla_x\Phi(x, y)) \times v(y)\,\mathrm{d}y \\ &= \operatorname{curl} \int_B \operatorname{curl}_x(\Phi(x, y)v(y))\,\mathrm{d}y \\ &= \operatorname{curl} \operatorname{curl} \int_B \Phi(x, y)v(y)\,\mathrm{d}y \end{aligned} \tag{3.10}$$

for $x \in B_e$. For the last step we will use the formula

$$\operatorname{div}(\Phi(x, y)v(y)) = \nabla\Phi(x, y) \cdot v(y) + \Phi(x, y) \underbrace{\operatorname{div} v(y)}_{=0} \tag{3.11}$$

for both differentiations with respect to x and y and an application of Gauss' theorem which yields

$$\int_B \operatorname{div}_y(\Phi(x, y)v(y))\,\mathrm{d}y = \int_{\partial B} \Phi(x, y) \underbrace{\nu \cdot v(y)}_{=0 \text{ on } \partial B} \mathrm{d}s(y) = 0. \tag{3.12}$$

Employing curl curl $= -\triangle + \text{grad div}$ and $\triangle_x \Phi(x, y) = 0$ for $y \in B$, $x \in B_e$ we proceed as follows:

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\int_B \Phi(x,y)v(y)\,\mathrm{d}y &= (-\triangle + \operatorname{grad}\operatorname{div})\int_B \Phi(x,y)v(y)\,\mathrm{d}y\\ &= \operatorname{grad}\int_B \operatorname{div}_x(\Phi(x,y)v(y))\,\mathrm{d}y\\ &\overset{(3.11)}{=} \operatorname{grad}\int_B (\nabla_x\Phi(x,y))\cdot v(y)\,\mathrm{d}y \qquad (3.13)\\ &= -\operatorname{grad}\int_B (\nabla_y\Phi(x,y))\cdot v(y)\,\mathrm{d}y\\ &\overset{(3.11)}{=} -\operatorname{grad}\int_B \operatorname{div}_y(\Phi(x,y)v(y))\,\mathrm{d}y\\ &\overset{(3.12)}{=} 0. \qquad (3.14)\end{aligned}$$

A combination of (3.9), (3.10) and (3.14) now concludes the proof. □

Our second goal of this section is to characterize the orthogonal complement of $N(W)$ with respect to the L^2-scalar product on B, i.e. we study the space

$$X^{\perp} := \{j \in H_{\text{div}=0}(B) \colon \langle j, j_0\rangle_{L^2(B)} = 0\ \forall j_0 \in X\}.$$

We start with the study of harmonic vector fields, i.e. fields v which satisfy $\operatorname{div} v = 0$ and $\operatorname{curl} v = 0$ in B.

Lemma 3.4. *Harmonic vector fields are a subset of* $X^{\perp}$, *i.e.*

$$\{j \in L^2(B) \colon \operatorname{curl} j = 0, \operatorname{div} j = 0\} \subset X^{\perp}. \qquad (3.15)$$

Proof. We need to show that for a field $j \in L^2(B)$ with $\operatorname{curl} j = 0$ and $\operatorname{div} j = 0$ the equation $\langle j, j_0\rangle = 0$ is satisfied for all $j_0 \in X$. We employ the formula

$$\operatorname{div}(a \times b) = b \cdot \operatorname{curl} a - a \cdot \operatorname{curl} b \qquad (3.16)$$

equations (3.1), (3.2) and Gauss' theorem to calculate

$$\begin{aligned}\langle j, j_0\rangle_{L^2(B)} &= \int_B j(y)\cdot j_0(y)\,\mathrm{d}y\\ &\overset{(3.2)}{=} \int_B j(y)\cdot \operatorname{curl}(Wj_0)(y)\,\mathrm{d}y\\ &= \int_B \{j(y)\cdot\operatorname{curl}(Wj_0)(y) - (Wj_0)(y)\underbrace{\operatorname{curl} j(y)}_{=0}\}\,\mathrm{d}y\\ &\overset{(3.16)}{=} \int_B \operatorname{div}(Wj_0(y)\times j)\,\mathrm{d}y\\ &= \int_{\partial B} \nu\cdot(Wj_0(y)\times j)\,\mathrm{d}s(y)\\ &\overset{(3.1)}{=} 0 \qquad (3.17)\end{aligned}$$

where we used the equation $\operatorname{curl} Wj_0 = j_0$ for $j_0 \in N(W)$ from (3.2). □

As a preparation for the final step of this section we need to study the orthogonal space of $H_0^1(B) \cap H_{\text{div}=0}(B)$.

Lemma 3.5. *If $f \in H^{-1}(B)$ satisfies*

$$\int_B f \cdot v \,\mathrm{d}x = 0 \qquad v \in H_0^1(B) \cap H_{\mathrm{div}=0}(B), \tag{3.18}$$

then there exists $q \in L^2(B)$ such that

$$f = \mathrm{grad}\, q. \tag{3.19}$$

Proof. For a proof we refer to [6], lemma 2.1. □

We are now prepared to prove our second main result. To this end we define

$$Y := \{j \in H_{\mathrm{div}=0}(B) \colon \exists q \in L^2(B) \text{ s.th. } \mathrm{curl}\, j = \mathrm{grad}\, q\}, \tag{3.20}$$

where the equation $\mathrm{curl}\, j = \mathrm{grad}\, q$ holds in $H^{-1}(B)$ in the sense

$$\int_B j \cdot \mathrm{curl}\, v \,\mathrm{d}x = \int_B q \,\mathrm{div}\, v \,\mathrm{d}x \qquad \forall\, v \in H_0^1(B). \tag{3.21}$$

Theorem 3.6. *The orthogonal space $N(W)^{\perp}$ is given by*

$$N(W)^{\perp} = Y. \tag{3.22}$$

Proof. First, we show the inclusion $Y \subset N(W)^{\perp}$. Consider elements $j \in Y$ and $j_0 \in N(W)$. We need to show that $\langle j, j_0\rangle_{L^2(B)} = 0$. By definition of Y there is $q \in L^2(B)$ such that (3.21) is satisfied. In particular, it is satisfied for $v := W j_0 \in H_0^1(B)$. Now, we calculate

$$\begin{aligned}\langle j, j_0\rangle_{L^2(B)} &= \int_B j(y) \cdot j_0(y) \,\mathrm{d}y \\ &\overset{\text{lemma 3.2}}{=} \int_B j(y) \cdot \mathrm{curl}(W j_0)(y) \,\mathrm{d}y \\ &\overset{(3.21)}{=} \int_B q(y) \,\mathrm{div}(W j_0)(y) \,\mathrm{d}y \\ &\overset{(2.15)}{=} 0,\end{aligned} \tag{3.23}$$

which proves the first part of the statement.

Second, we show that the inclusion $N(W)^{\perp} \subset Y$ is satisfied. For $j \in N(W)^{\perp}$ we need to show that $j \in Y$. Consider an element $v \in H_0^1(B) \cap H_{\mathrm{div}=0}(B)$. Then $\mathrm{curl}\, v$ is an element of $X = N(W)$. Now, Gauss' theorem in combination with (3.16) yields

$$\int_B v \cdot \mathrm{curl}\, j \,\mathrm{d}x = \int_B j \cdot \mathrm{curl}\, v \,\mathrm{d}x = 0. \tag{3.24}$$

Finally, an application of lemma 3.5 to $f := \mathrm{curl}\, j$ yields the existence of $q \in L^2(B)$ such that $\mathrm{curl}\, j = \mathrm{grad}\, q$, i.e. $j \in Y$ and the proof is complete. □

Theorems 3.3 and 3.6 yield a decomposition of the space $H_{\mathrm{div}=0}(B)$ as follows.

Corollary 3.7. *We have the orthogonal decomposition*

$$H_{\mathrm{div}=0}(B) = N(W) \oplus N(W)^{\perp} = X \oplus Y. \tag{3.25}$$

4. The nullspace $N(W)$ of W by example and uniqueness for discrete wire networks

First, the goal of this final section is to provide examples for elements in $N(W)$ and an example for elements which are not in $N(W)$. Second, we will prove uniqueness for current reconstructions in the case of discrete wire networks. We start with the non-nullspace example.

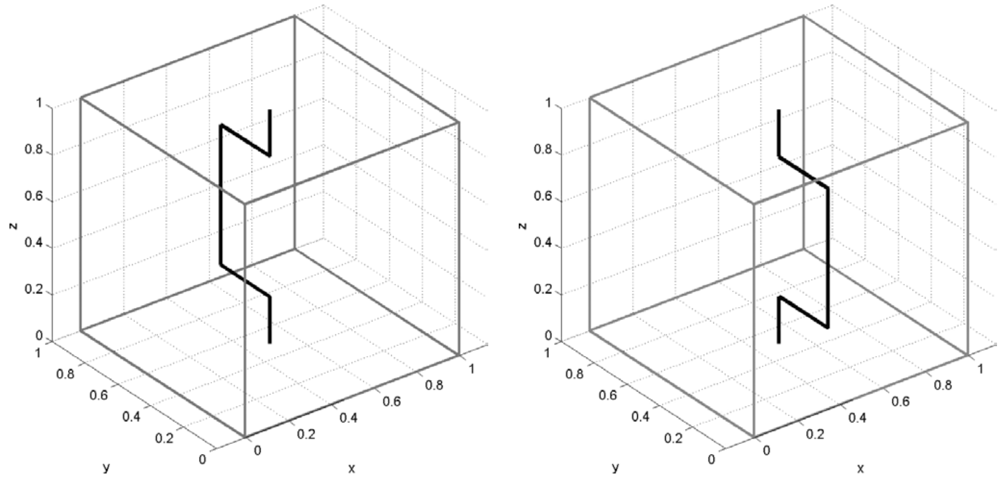

Figure 2. Example for two particular distributions of the conductivity which leads to currents around the black lines.

4.1. Non-nullspace example

We would like to show that not all divergence-free fields $j \in H_{\mathrm{div}=0}(B)$ with $\nu \cdot j = 0$ generate a vanishing magnetic field $H = Wj$ outside of B. Let B be a cuboid in $\mathbb{R}^3$ and $\Gamma_j : [0, 1] \to B$, $j = 1, 2$ as indicated by the black lines in figure 2. We assume that a sufficiently smooth conductivity distribution $\sigma(x)$ is equal to σ_1 in a neighbourhood

$$V(h, j) := \{x\colon d(x, \gamma_j) < h\} \tag{4.1}$$

of size $h > 0$ of the wires Γ_1 or Γ_2, respectively, and equal to σ_0 outside of $V(2h, j)$, $j = 1, 2$. Consider a constant current injection flowing from the centre of the bottom square to the centre of the top square for both settings. The difference of the two settings yields a current distribution j depending on σ_0, σ_1 and h which satisfies

$$\operatorname{div} j = 0 \quad \text{in } B, \qquad \nu \cdot j = 0 \quad \text{on } \partial B. \tag{4.2}$$

For $h \to 0$, $\sigma_0 \to 0$ and constant σ_1 the currents j tend to a simple quadratic wire loop which does generate a non-vanishing magnetic field outside of B, i.e. for sufficiently small h and sufficiently small σ_0 the element j with $\nu \cdot j = 0$ on ∂B is not in the nullspace $N(W)$ of the operator W.

4.2. Nullspace example

As a second example we would like to construct particular elements of the nullspace $N(W)$ of W and study their properties. Consider a closed current loop defined by

$$\Gamma(t) := r_0 \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} - r_1 \cos(a \cdot t) \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} + r_1 \sin(a \cdot t) \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{4.3}$$

with the two radii r_0 and r_1 such that $r_1 < r_0$. An example with $r_0 = 5$ and $r_1 = 2$ is illustrated in figure 3. The current I in the wire is chosen proportional to $1/a$ when the number of turns a is increased. Then the surface current density for the torus is converging towards a constant surface current density.

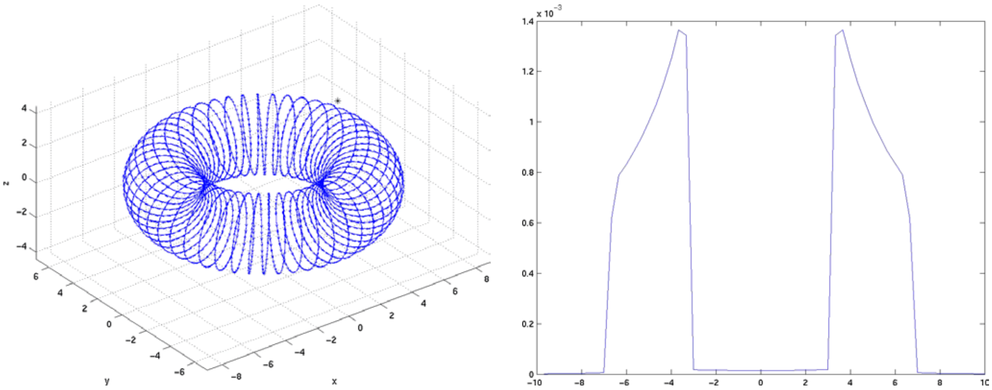

Figure 3. Current loop which in the limit for infinite number of turns and constant current density on the boundary of the torus is an element of the nullspace $N(W)$. The right image shows the norm of the simulated magnetic field on the line from $(-10, 0, 1)$ to $(10, 0, 1)$ which crosses the torus twice for the x-coordinate approximately between -7 and -3 and 3 and 7. For the left image we used 2000 sampling points and 50 turns, the right image is calculated with 10 000 sampling points and $a = 200$ turns.

The loop is the discretized version of a nullspace element j_0 which is the curl of a vector field v flowing in a homogeneous way through the torus. The curl of v is zero inside of the torus and outside of the torus. On the boundary only the derivative in the direction of the normal is non-zero. For a boundary point x of the torus we choose a coordinate system where $\nu(x)$ coincides with the negative x-axis such that the direction of the flow v in x is the unit vector $e_3 = (0, 0, 1)^T$. Then, we obtain

$$j(x) = \operatorname{curl} v(x) = \operatorname{curl} \begin{pmatrix} 0 \\ 0 \\ v_3(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\partial_x v_3(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta(x) v_3(x) \\ 0 \end{pmatrix} \tag{4.4}$$

with the size $v_3(x)$ of the jump of v at x. If the loop is placed in the interior of the domain B, then it satisfies $\operatorname{div} v = 0$ and the boundary condition $\gamma_0[B]v = 0$, i.e. $j = \operatorname{curl} v$ is an element of $X = N(W)$.

4.3. Different fuel cell currents with the same magnetic field

Sometimes directional constraints of the form

$$\mathbf{e}_3 \cdot j(y) \geqslant 0 \qquad \forall\, y \in B \tag{4.5}$$

with $\mathbf{e}_3 := (0, 0, 1)^T$, which come from the particular design and chemistry of fuel cells, are introduced as additional 'boundary conditions'. However, this does solve the problem with the nullspace elements only partly.

Of course, j_0 does not satisfy directional constraints. Further, we know from [4] that elements of the nullspace, which satisfy directional constraints, must be zero. Thus, any element of $N(W)$ will not satisfy estimate (4.5).

However, consider any homogeneous current density j_{hom} flowing through a domain B in the z-direction, i.e. it satisfies the directional constraint (4.5). Then, we might construct

$$j_\beta := j_{\text{hom}} + \beta j_0 \tag{4.6}$$

with some parameter $\beta \in \mathbb{R}$. For all sufficiently small β the current density j_β also satisfies the directional constraints (4.5). Further, we calculate

$$W j_\beta = W j_{\text{hom}} + \beta \underbrace{W j_0}_{=0} = W j_{\text{hom}}, \tag{4.7}$$

i.e. all current densities j_β produce the same magnetic field outside the domain B. Thus, it is not possible to decide from the measurement of the exterior magnetic field H which admissible current density j_β has generated the measurements. The problem arises both for fuel cell stacks and for single cells.

The introduction of an upper limit for the current density is another possible constraint which leads to a natural method of stabilization of the problem. However, it does not influence the nullspace question.

In practical experiments with fuel cells [3] the above nullspace element (i.e. a hot spot at the centre surrounded by a ring of low current density) has not been observed yet. Why is this the case? The answer is the high degree of discretization which is currently used in the numerical simulation of experiments. In a cell with a discretization of $6 \times 6 \times 3$ the nullspace element cannot be realized. As our final result we will show that discrete wire models will always lead to uniqueness for current reconstructions.

4.4. Uniqueness for discrete wire networks

Finally, we show that for wire grids the nullspace is trivial, i.e. we have uniqueness for the current reconstruction.

Consider a wire network N, which we consider to be a collection of discrete straight wires $l = \overline{y_1 y_2}$ from y_1 to y_2 with different conductivity σ which are connected only at their end points y_1 or y_2, respectively. Currents may flow through the wires from some point x_1 to some other point x_2. For difference reconstructions we assume that the wire network is a closed system. We note that the currents j through a wire network are not an element of $L^2(B)$ and that the theory of the first sections does not apply to this situation.

As a preparation we calculate the magnetic field H of some current flowing through the line l_0 from $y_1 = (-a, 0, 0)^T$ to $y_2 = (a, 0, 0)^T$, $a > 0$ in the point $x = (0, 0, h)^T$. We obtain

$$j(y) = \begin{pmatrix} j_x(y) \\ 0 \\ 0 \end{pmatrix}, \qquad |x - y| = \sqrt{y^2 + h^2}, \qquad (x - y) \times j(y) = \begin{pmatrix} 0 \\ h j_x(y) \\ 0 \end{pmatrix}, \tag{4.8}$$

where j_x denotes the x-component of the current, and thus

$$\begin{aligned} H(x) &= \frac{-1}{4\pi} \int_{l_0} \frac{x - y}{|x - y|^3} \times j(y)\, \mathrm{d}l(y) \\ &= \frac{-h j_x}{4\pi} \int_{-a}^{a} \frac{1}{(y^2 + h^2)^{\frac{3}{2}}}\, \mathrm{d}y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{-h j_x}{4\pi} \frac{2a}{h^2 \sqrt{a^2 + h^2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{j_x}{2\pi h} \frac{a}{\sqrt{a^2 + h^2}} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.9}$$

For $h \to 0$ the magnetic field has a singularity of order $1/h$. Now, we obtain the following uniqueness result.

Theorem 4.1. *Let j be a current flowing through our wire network which is located in a domain B such that no current is flowing through the boundaries of B, i.e. $\nu \cdot j = 0$ on ∂B. We assume that there are no sources or sinks inside the network, i.e. the current j satisfies* $\mathrm{div}\, j = 0$. *If the magnetic field H generated by the current j vanishes in the exterior B_e of B, then we obtain $j = 0$ in N.*

Proof. The magnetic field is an analytic function in the exterior of N. By assumption it is zero in B_e, thus it is zero in N_e. In particular, it is zero in a neighbourhood of each discrete wire l. However, any current flowing through l would generate a singularity of H at l, compare equation (4.9), where the calculation is carried out for a particular choice of the coordinate system. Thus, the current must vanish in l and by the same argument in each part l of the whole network N. This ends the proof. □

5. Conclusions

We have characterized the nullspace of the Biot–Savart operator in theorems 3.3 and 3.6. This is an important step for the understanding of the magnetic tomography problem and a basis for the development of efficient algorithms for current reconstructions which take into account the particular form of the orthogonal complement $N(W)^{\perp}$ of the nullspace $N(W)$.

Further, a particular element of the nullspace has been constructed and the behaviour of the magnetic field has been numerically verified for a discretization of this element. Finally, we have shown that the magnetic tomography problem for discrete wire networks is always uniquely solvable.

References

[1] Colton D and Kress R 1992 *Inverse Acoustic and Electromagnetic Scattering Theory* (Berlin: Springer)

[2] Costabel M 1988 Boundary integral operator on Lipschitz domains: elementary results *SIAM J. Math. Anal.* **19** 613–26

[3] Hauer K H, Potthast R, Stolten D and Wüster T 2005 Magnetotomography—a new method for analysing fuel cell performance and quality *J. Power Sources* at press

[4] Kress R, Kühn L and Potthast R 2002 Reconstruction of a current distribution from its magnetic field *Inverse Problems* **18** 1127–46

[5] Potthast R and Kühn L 2003 On the convergence of the finite integration technique for the anisotropic boundary value problem of magnetic tomography *Math. Methods Appl. Sci.* **26** 739–57

[6] Girault V and Raviart P-A 1986 *Finite Element Approximation of the Navier Stokes Equations* (Berlin: Springer)